Abstract Algebra
Day 24 Class Work

Here's the group table for D_4. Recall that for σ, $\tau \in D_4$, the "product" $\sigma \circ \tau$ is the entry in row σ and column τ. For example, the product $d \circ r_{90} = v$ is shown in bold. ← You'll be using this a lot.

$\circ$	ε	$\mathbf{r_{90}}$	r_{180}	r_{270}	h	v	d	d'
ε	ε	r_{90}	r_{180}	r_{270}	h	v	d	d'
r_{90}	r_{90}	r_{180}	r_{270}	ε	d'	d	h	v
r_{180}	r_{180}	r_{270}	ε	r_{90}	v	h	d'	d
r_{270}	r_{270}	ε	r_{90}	r_{180}	d	d'	v	h
h	h	d	v	d'	ε	r_{180}	r_{90}	r_{270}
v	v	d'	h	d	r_{180}	ε	r_{270}	r_{90}
$\mathbf{d}$	d	$\mathbf{v}$	d'	h	r_{270}	r_{90}	ε	r_{180}
d'	d'	h	d	v	r_{90}	r_{270}	r_{180}	ε

1. We'll analyze how the shortcut *fails*. Let $H = \{\varepsilon, v\}$ be a subgroup of D_4.

   (a) Compute the cosets $r_{270}H$ and dH. **Ans:** $r_{270}H = \{r_{270}, d'\}$.

   (b) Compute $r_{270}H \cdot dH$ by multiplying each element of $r_{270}H$ by those of dH.

   $$r_{270}H \cdot dH = \{r_{270}, d'\} \cdot \{d, r_{90}\}$$
   $$= \{ \qquad \qquad \qquad \qquad \}$$
   $$= \{ \qquad \qquad \qquad \qquad \}$$

   Before proceeding, verify that you found $r_{270}H \cdot dH = \{v, \varepsilon, r_{180}, h\}$.

   (c) Anita says, "The product $r_{270}H \cdot dH$ is definitely *not* a coset of H, because it has too many elements." What might she mean? ← The CM shortcut fails!

   (d) Compute $(r_{270} \cdot d)H$ and verify that $(r_{270} \cdot d)H \subseteq r_{270}H \cdot dH$. **Ans:** $(r_{270} \cdot d)H = \{v, \varepsilon\}$.

   Note: In fact, this inclusion *always* holds. (See Problem #2.)

2. **Prove:** Let G be a group and H a subgroup. For $a, b \in G$, define the coset product by

   $$aH \cdot bH = \{\alpha \cdot \beta \mid \alpha \in aH,\ \beta \in bH\}.$$

   Then $(ab)H \subseteq aH \cdot bH$. (**Hint:** Let $x \in (ab)H$ and show that $x \in aH \cdot bH$.) **Ans:** $x = (ab)h = a\varepsilon \cdot bh$.

What we know so far:

Since $(ab)H \subseteq aH \cdot bH$ always holds, the key to coset multiplication is $aH \cdot bH \subseteq (ab)H$. When does *this* set inclusion hold? We'll explore in the next few questions!

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3. Let G be a *commutative* group, H a subgroup, and $a, b \in G$.

   (a) Prove that $aH \cdot bH \subseteq (ab)H$.

   Hint: Let $\alpha\beta \in aH \cdot bH$, where $\alpha \in aH$ and $\beta \in bH$. Show that $\alpha\beta \in (ab)H$. ← So, α and β look like...

   (b) Where in your proof in part (a) did you use the fact that G is commutative?

4. Let $K = \{\varepsilon, r_{90}, r_{180}, r_{270}\}$ be a subgroup of D_4.

   (a) For each $a \in D_4$, compute the left coset aK and the right coset Ka.

   Hint: What if a is a rotation? What if it's a reflection? ← You shouldn't have to do much calculation here.

   (b) **True or False:** $aK = Ka$ for all $a \in D_4$.

   (c) **True or False:** $aK = Ka$ means $ak = ka$ for each $k \in K$.

   Note: In other words, does coset equality imply element-by-element equality?

5. Consider again the subgroup $K = \{\varepsilon, r_{90}, r_{180}, r_{270}\}$ of D_4. In this problem, we'll analyze why the set inclusion $dK \cdot vK \subseteq (dv)K$ must hold.

   (a) Find the element $\boxed{?} \in K$ such that $r_{90}v = v\boxed{?}$. **Ans:** $\boxed{?} = r_{270}$.

   (b) Let $\alpha\beta \in dK \cdot vK$, where $\alpha = dr_{90} \in dK$ and $\beta = vr_{180} \in vK$. Using only your result from part (a), *and without looking at the D_4 table again*, explain why $\alpha\beta \in (dv)K$.

   Hint: The argument here should be similar to Problem #3(a). ← But how is it different?

   (c) Explain why $dK \cdot vK \subseteq (dv)K$. Your work from Problem #4 should help.

6. **(If you were comfortable with Problem #5, feel free to skip this one!)**

   Consider yet again the subgroup $K = \{\varepsilon, r_{90}, r_{180}, r_{270}\}$ of D_4. In this problem, we'll analyze why the set inclusion $hK \cdot d'K \subseteq (hd')K$ must hold.

   (a) Find the element $\boxed{?} \in K$ such that $r_{270}d' = d'\boxed{?}$. **Ans:** $\boxed{?} = r_{90}$.

   (b) Let $\alpha\beta \in hK \cdot d'K$, where $\alpha = hr_{270} \in hK$ and $\beta = d'r_{90} \in d'K$. Using only your result from part (a), explain why $\alpha\beta \in (hd')K$. Don't look at the D_4 table again!

   (c) Explain why $hK \cdot d'K \subseteq (hd')K$. Your work from Problem #4 should help.

Definition. Let H be a subgroup of a group G. Then H is called a *normal subgroup* of G if $gH = Hg$ for all $g \in G$.

- In other words, all left and right cosets of H are equal.
- We often say, "H is normal in G."

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7. Suppose H is a normal subgroup of G, and let $a, b \in G$.

   (a) Let $hb \in Hb$ for some h in H. Elizabeth says, "Since $Hb = bH$, we must have $hb = bh$." How would you correct her claim?

   (b) Prove that $aH \cdot bH \subseteq (ab)H$. You may *not* assume that G is commutative.

   Hint: $ah \cdot bk = a(hb)k = \cdots$

8. (a) Explain why the subgroup $H = \{\varepsilon, v\}$ is *not* normal in D_4.

   (b) Explain why the subgroup $Z = \{\varepsilon, r_{180}\}$ is normal in D_4.

   Hint: Z is the *center* of D_4, which means. . . ?

   (c) **True or False:** If G is commutative, then every subgroup H is normal in G. ← Explain your reasoning.

9. Let N be a normal subgroup of G and let H be any subgroup of G. Define the set product $NH = \{nh \mid n \in N, h \in H\}$. Prove that NH is a subgroup of G.