Abstract Algebra
Day 23 Class Work

Unless specified otherwise, assume the shortcut holds in G/H.

← i.e., $aH \cdot bH = (ab)H$.

1. G/H is a group. Therefore, any property that we know about groups applies to G/H. Let's consider the "socks-shoes," for example. What goes into the empty boxes?

$$(aH \cdot bH)^{-1} = \boxed{}H \cdot \boxed{}H.$$

2. Let G be a commutative group and H its subgroup. Prove that G/H is commutative.

   Hint: Start with two elements of G/H. What do those elements look like? **Ans:** aH and bH.

3. Consider the subgroup $H = \{1, 3, 9\}$ of U_{13}.

   (a) Find the order of 4 in U_{13}. **Ans:** $\text{ord}(4) = 6$.

   (b) Find the order of $4H$ in U_{13}/H.

   (c) Anita claims, "If $\text{ord}(4) = 6$, then $(4H)^6 = 4^6H = 1H$. Thus, $\text{ord}(4H) = 6$, too." Fix the error in her argument.

4. **Prove:** Let $a \in G$ with finite order. Then $\text{ord}(aH)$ in G/H is a divisor of $\text{ord}(a)$ in G. ← Anita's error is quite useful in this proof.

   Hint: In a group, say $(\text{blah})^{10}$ equals the identity. What can you say about $\text{ord}(\text{blah})$?

5. Consider the statement:

   If $aH = bH$ in G/H, then $a = b$ in G.

   Is it true or false? If it's true, prove it. If it's false, give a counter-example.

6. **Prove:** $(gH)^n = \varepsilon H$ if and only if $g^n \in H$. **Hint:** $aH = H \Leftrightarrow a \in H$.

7. Suppose $[G : H] = n$. Show that $g^n \in H$ for all $g \in G$.

   Recall: $[G : H]$ is the number of (left) cosets of H, which is also the size of G/H.

   Hint: Consider the element $gH \in G/H$, and let $d = \text{ord}(gH)$. How are d and n related?

8. Let G be a group and $Z = \{z \in G \mid zg = gz \text{ for all } g \in G\}$. Prove that the shortcut holds in G/Z. In other words, given $aZ, bZ \in G/Z$, define the coset product by ← i.e., Z is the *center* of G.

$$aZ \cdot bZ = \{\alpha \cdot \beta \mid \alpha \in aZ,\ \beta \in bZ\}.$$

   Then show that $aZ \cdot bZ = (ab)Z$.

   Note: This is a set equality proof. You must show $aZ \cdot bZ \subseteq (ab)Z$ and $(ab)Z \subseteq aZ \cdot bZ$.

9. Let G be a group and H a subgroup of G. Determine if each statement is true or false. If it's true, prove it. If it's false, give a counterexample.

   (a) If G and H are finite, then G/H is finite.

   (b) If G/H is finite, then G and H are finite.

   (c) If G is infinite and H is finite, then G/H is infinite.

10. (a) Find an additive group G, a subgroup H, and an element $a \in G$ such that:

$$a + H \neq 0 + H,\ \operatorname{ord}(a + H) \text{ in } G/H \text{ is finite, and } \operatorname{ord}(a) \text{ in } G \text{ is infinite.}$$

    (b) Same as part (a), but this time, find a commutative, multiplicative group.

11. If G/H has an element of order n, show that G has an element of order n.

    ← Assume that each $g \in G$ has finite order.

    Hint: Let $gH \in G/H$ be an element of order n. Now, come up with an element in G with order n. An earlier problem might come in handy.

12. Let G be a group, and define Z as in Problem #8.

    Prove: If G/Z is cyclic, then G is commutative.