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Abstract Algebra
Day 22 Class Work

Below, we'll consider the subgroup $H = \langle 10 \rangle$ of the (multiplicative) group U_{37}.

1. (a) Quick, how many elements are in U_{37}? ← Yes, 37 is prime.

   (b) Verify that $\langle 10 \rangle = \{1,\ 10,\ 26\}$ by computing the powers of 10 modulo 37.

   (c) How many distinct cosets of H are there? How do you know? **Ans:** 12 cosets.

2. **Suggestion:** Use a calculator for parts (a) and (b) below.

   (a) Compute the cosets $4H$ and $11H$. **Ans:** $4H = \{4,\ 3,\ 30\}$.

   (b) Compute the coset product $4H \cdot 11H$ *without using the coset multiplication shortcut*, and verify that it is, indeed, equal to $44H$.

   Note: It might help to write $4H = \{4,\ 3,\ -7\}$ and $11H = \{11,\ -1,\ -10\}$. (Why?)

You may use the coset multiplication shortcut for the rest of the problems!

← i.e., $aH \cdot bH = (ab)H$.

3. Find the coset product $1H \cdot 11H$. (**Optional:** Compute this *without* using the shortcut.) What does this say about the role of $1H$ in U_{37}/H?

4. (a) Anita claims that $2H$ and $19H$ in U_{37}/H are inverses of each other. Do you agree or disagree with her? Explain.

   (b) Find the inverse of $15H$ in U_{37}/H. How about the inverse of $28H$?

5. Verify that U_{37}/H with coset multiplication satisfies the group properties. Note that...

   - Since U_{37} is commutative, the shortcut $aH \cdot bH = (ab)H$ holds in U_{37}/H.
   - You may assume that coset multiplication is associative. ← Proved in Chapter 21.

6. (a) Find all $a \in U_{37}$ such that $aH = 1H$.

   (b) Find the order of $6H$ in U_{37}/H. **Ans:** $\text{ord}(6H) = 4$.

   Note: Feel free to use the shortcut to compute the powers of $6H$.

   (c) Verify that $\text{ord}(34H) = 3$ in U_{37}/H. It might help to write $34H = (-3)H$.

   (d) Find the order of $4H$ in U_{37}/H. **Ans:** $\text{ord}(4H) = 6$.

   (e) In U_{37}, it turns out that: $\text{ord}(6) = 4$, $\text{ord}(34) = 9$, $\text{ord}(4) = 18$. Any conjectures?

7. (a) Let $a \in U_{37}$ with $\text{ord}(a) = 12$. Show that $(aH)^{12} = 1H$. What does this say about the order of aH in U_{37}/H?

   (b) **Prove:** $\text{ord}(aH)$ in U_{37}/H is a divisor of $\text{ord}(a)$ in U_{37} for all $a \in U_{37}$.

8. Recall that U_{13} is cyclic with generator 2, i.e., $U_{13} = \langle 2 \rangle$. With the subgroup $H = \{1,\ 3,\ 9\}$ of U_{13}, verify that U_{13}/H is cyclic with generator $2H$.

9. Note that $\mathbb{Z}_{12}$ is cyclic with generator 1, i.e., $\mathbb{Z}_{12} = \langle 1 \rangle$. With the subgroup $H = \{0,\ 4,\ 8\}$ of $\mathbb{Z}_{12}$, verify that $\mathbb{Z}_{12}/H$ is cyclic with generator $1 + H$.

10. Consider the subgroup $H = \{1, 7\}$ of U_{16}.
    (a) Quick, how many distinct cosets of H are there? Explain how you know.
    (b) Find the quotient group U_{16}/H.
    (c) Create the table for U_{16}/H and verify that it's a group under coset multiplication.
    (d) Find the order of each $aH \in U_{16}/H$. Is the group cyclic?

11. Consider the subgroup $H = \{1, 9\}$ of U_{16}.
    (a) Find the quotient group U_{16}/H and determine if it's cyclic.
    (b) Compare your work in part (a) with Problem #10. Are you surprised by the results?
    (c) U_{16} has another 2-element subgroup K. Find it and determine if U_{16}/K is cyclic.

12. **(An Important Proof)** Let G be a commutative group, H its subgroup, and $a, b \in G$. Define the coset product by

$$aH \cdot bH = \{\alpha \cdot \beta \mid \alpha \in aH,\ \beta \in bH\}.$$

    Then show that $aH \cdot bH = (ab)H$.

    ← i.e., The shortcut holds when G is commutative.

13. **(Some Food for Thought)** What about non-commutative groups? In Chapter 21 Exercises, we considered these subgroups of D_4: $Z = \{\varepsilon, r_{180}\}$ and $H = \{\varepsilon, v\}$. We found that D_4/Z satisfied the shortcut, but D_4/H did not. Can you explain what's going on?