Abstract Algebra
Day 1 Class Work Solutions

1. Which of these are true?

   (a) If I live in Tokyo, then I live in Japan.

   Solution. True.

   (b) If I live in Japan, then I live in Tokyo.

   Hint: I grew up in Osaka.

   Solution. False. I might live in Osaka, for instance.

   (c) If I don't live in Tokyo, then I don't live in Japan.

   Solution. False. I might live in Osaka, for instance.

   (d) If I don't live in Japan, then I don't live in Tokyo.

   Solution. True. (Do you see why?)

2. **Prove:** If n is an integer, then $n^2 + n$ is even.

   Hint: What if n is odd? What if n is even?

   PROOF. Assume n is an integer. We consider the two cases: (1) n is odd and (2) n is even.
   Case (1): Suppose n is odd, so that $n = 2k + 1$ for some integer k. Then

   $$n^2 + n = (2k + 1)^2 + (2k + 1) = 4k^2 + 6k + 2 = 2(2k^2 + 3k + 1),$$

   where $2k^2 + 3k + 1$ is an integer. Thus, $n^2 + n$ is even.

   Case (2): Suppose n is even, so that $n = 2k$ for some integer k. Then

   $$n^2 + n = (2k)^2 + 2k = 4k^2 + 2k = 2(2k^2 + k),$$

   where $2k^2 + k$ is an integer. Thus, $n^2 + n$ is even. ■

3. **Prove:** If n^2 is odd, then n is odd. (Here, n is an integer.)

   Hint: Setting $n^2 = 2k + 1$ is *not* a good idea. (Why not?)

   Note: We will prove its contrapositive, namely: *If n is even, then n^2 is even.*

   PROOF. Assume n is even. Thus, $n = 2k$ for some integer k. Then, $n^2 = (2k)^2 = 2(2k^2)$, where $2k^2$ is an integer. Therefore, n^2 is even. ■

4. Consider the statement:

   If n is an odd integer, then n is a prime number.

   Which value of n is a *counterexample* showing that the statement is false? Explain.

   ← Just one counterexample is needed to show that a statement is false.

   - $n = 13$
   - $n = 15$

   Solution. $n = 15$ is a counterexample. Although $n = 15$ is odd (i.e., the hypothesis is true), it is *not* a prime number (i.e., the conclusion is false).

5. Consider the statement:

   If n is a prime number, then $n + 2$ is also a prime number.

   Find a counterexample to show that the statement is false.

Solution. $n = 13$ is a counterexample. Although $n = 13$ is prime (i.e., the hypothesis is true), $n + 2 = 15$ is *not* a prime number (i.e., the conclusion is false).

6. Consider the following statements:

   - If n is prime, then $2^n - 1$ is prime.
   - If $2^n - 1$ is prime, then n is prime.

   Determine whether or not each statement is true. If it's true, prove it. If it's false, come up with a counterexample.

   Hint: Start by trying out some concrete values of n.