Abstract Algebra
Day 19 Class Work Solutions

1. Consider the (multiplicative) group U_{13} and its subgroup $H = \{1, 3, 9\}$. We just saw an example of a *coset* of H, namely $6H = \{6 \cdot 1,\ 6 \cdot 3,\ 6 \cdot 9\} = \{6, 5, 2\}$.

   (a) For each $a \in U_{13}$, compute the coset aH, i.e., compute $1H$, $2H$, $3H$, $\ldots$, $12H$.

   Suggestion: Split the work among your table members.

   Solution.

$$
\begin{aligned}
1H &= \{1 \cdot 1, 1 \cdot 3, 1 \cdot 9\} = \{1, 3, 9\} & 7H &= \{7 \cdot 1, 7 \cdot 3, 7 \cdot 9\} = \{7, 8, 11\} \\
2H &= \{2 \cdot 1, 2 \cdot 3, 2 \cdot 9\} = \{2, 6, 5\} & 8H &= \{8 \cdot 1, 8 \cdot 3, 8 \cdot 9\} = \{8, 11, 7\} \\
3H &= \{3 \cdot 1, 3 \cdot 3, 3 \cdot 9\} = \{3, 9, 1\} & 9H &= \{9 \cdot 1, 9 \cdot 3, 9 \cdot 9\} = \{9, 1, 3\} \\
4H &= \{4 \cdot 1, 4 \cdot 3, 4 \cdot 9\} = \{4, 12, 10\} & 10H &= \{10 \cdot 1, 10 \cdot 3, 10 \cdot 9\} = \{10, 4, 12\} \\
5H &= \{5 \cdot 1, 5 \cdot 3, 5 \cdot 9\} = \{5, 2, 6\} & 11H &= \{11 \cdot 1, 11 \cdot 3, 11 \cdot 9\} = \{11, 7, 8\} \\
6H &= \{6 \cdot 1, 6 \cdot 3, 6 \cdot 9\} = \{6, 5, 2\} & 12H &= \{12 \cdot 1, 12 \cdot 3, 12 \cdot 9\} = \{12, 10, 4\}
\end{aligned}
$$

   (b) How many *distinct* cosets did you find?

   Ans: Less than 12.

   Solution. There are four distinct cosets:

   - $1H = 3H = 9H = \{1, 3, 9\}$ (original subgroup)
   - $2H = 5H = 6H = \{2, 5, 6\}$
   - $4H = 10H = 12H = \{4, 10, 12\}$
   - $7H = 8H = 11H = \{7, 8, 11\}$

   (c) Look ahead to Problem #5 to verify that you've found the cosets correctly.

   (d) Write down any observations you have about these cosets.

   Solution. Answer will vary. See Section 19.1 in the textbook for details.

2. Consider the (additive) group $\mathbb{Z}_{12}$ and its subgroup $H = \{0, 4, 8\}$. We saw an example of a coset of H, namely $6 + H = \{6 + 0,\ 6 + 4,\ 6 + 8\} = \{6, 10, 2\}$.

   ← It's $6 + H$ instead of $6H$ for an additive group.

   (a) How many *distinct* cosets do you expect to find? How do you know?

   Solution. There are 12 and 3 elements in $\mathbb{Z}_{12}$ and H, respectively. So, there should be $12 \div 3 = 4$ distinct cosets of H.

   (b) For each $a \in \mathbb{Z}_{12}$, compute the coset $a + H$.

   ← Again, split the work among table members.

   Solution. As conjectured in part (a), we found four distinct cosets:

   - $0 + H = 4 + H = 8 + H = \{0, 4, 8\}$ (original subgroup)
   - $1 + H = 5 + H = 9 + H = \{1, 5, 9\}$
   - $2 + H = 6 + H = 10 + H = \{2, 6, 10\}$
   - $3 + H = 7 + H = 11 + H = \{3, 7, 11\}$

   (c) Again, look ahead to Problem #5 to check your cosets.

   (d) Write down any observations you have about these cosets.

   Solution. Answer will vary. See Section 19.2 in the textbook for details.

3. Consider the (multiplicative) group D_4 and its subgroup $H = \{\varepsilon, v\}$.

   (a) How many *distinct* cosets do you expect? Explain your reasoning.

   Solution. There are 8 and 2 elements in D_4 and H, respectively. So, there should be $8 \div 2 = 4$ distinct cosets of H.

   (b) For each $a \in D_4$, compute the coset aH. The following table should help.

   Remark: For $\sigma, \tau \in D_4$, the "product" $\sigma \circ \tau$ is the entry in row σ and column τ. For example, the product $d \circ r_{90} = v$ is shown in bold.

$\circ$	ε	$\boldsymbol{r_{90}}$	r_{180}	r_{270}	h	v	d	d'
ε	ε	r_{90}	r_{180}	r_{270}	h	v	d	d'
r_{90}	r_{90}	r_{180}	r_{270}	ε	d'	d	h	v
r_{180}	r_{180}	r_{270}	ε	r_{90}	v	h	d'	d
r_{270}	r_{270}	ε	r_{90}	r_{180}	d	d'	v	h
h	h	d	v	d'	ε	r_{180}	r_{90}	r_{270}
v	v	d'	h	d	r_{180}	ε	r_{270}	r_{90}
$\boldsymbol{d}$	d	$\boldsymbol{v}$	d'	h	r_{270}	r_{90}	ε	r_{180}
d'	d'	h	d	v	r_{90}	r_{270}	r_{180}	ε

   Solution. Indeed, there are 4 distinct left cosets of H, as shown below:

   - $\varepsilon H = vH = \{\varepsilon, v\}$
   - $r_{90}H = dH = \{r_{90}, d\}$
   - $r_{180}H = hH = \{r_{180}, h\}$
   - $r_{270}H = d'H = \{r_{270}, d'\}$

4. Come up with your own *multiplicative* group G and its subgroup H. ← Make sure G is finite.

   (a) How many distinct cosets do you expect to find?

   (b) Find all distinct cosets aH.

5. Here are some data from Problems #1 and #2, which you might find useful. ← You're welcome.

   Cosets of $H = \{1, 3, 9\}$ in U_{13} are...

   - $1H = 3H = 9H = \{1, 3, 9\}$
   - $2H = 5H = 6H = \{2, 5, 6\}$
   - $4H = 10H = 12H = \{4, 10, 12\}$
   - $7H = 8H = 11H = \{7, 8, 11\}$

   Cosets of $H = \{0, 4, 8\}$ in $\mathbb{Z}_{12}$ are...

   - $0 + H = 4 + H = 8 + H = \{0, 4, 8\}$
   - $1 + H = 5 + H = 9 + H = \{1, 5, 9\}$
   - $2 + H = 6 + H = 10 + H = \{2, 6, 10\}$
   - $3 + H = 7 + H = 11 + H = \{3, 7, 11\}$

   Here, the cosets $3 + H$ and $11 + H$ are the same, even though $3 \neq 11$ in $\mathbb{Z}_{12}$. But could we have predicted that $3 + H = 11 + H$ *without* computing these cosets? Let's find out!

   (a) Let G be a group, H a subgroup, and $a, b \in G$. Based on the examples above, describe how a and b must be be related so that:

   - $a + H = b + H$ (for additive groups). **Hint:** What do you notice about $a - b$ and $b - a$?

     Note: For additive groups, the relationship between a and b must be additive.

   - $aH = bH$ (for multiplicative groups). ← For example, $2H = 6H$.

     Note: Here, a and b are related multiplicatively.

Solution.

- For additive groups, $a + H = b + H$ if and only if $a - b \in H$ (or $b - a \in H$).
- For multiplicative groups, $aH = bH$ if and only if $b^{-1}a \in H$ (or $a^{-1}b \in H$).

(b) Prove your conjecture from part (a). ← Write a proof for the multiplicative case.

Solution. See Theorem 19.16 in the textbook.

6. Let $H = \{\varepsilon, v\}$ be a subgroup of D_4. In Problem #3, you computed the *left* cosets aH.

(a) For each $a \in D_4$, compute the *right* coset $Ha = \{ha \mid h \in H\}$.

Solution. There are 4 distinct right cosets of H, as shown below:

- $H\varepsilon = Hv = \{\varepsilon, v\}$
- $Hr_{90} = Hd' = \{r_{90}, d'\}$
- $Hr_{180} = Hh = \{r_{180}, h\}$
- $Hr_{270} = Hd = \{r_{270}, d\}$

(b) **True or False:** $aH = Ha$ for all $a \in D_4$.

Solution. False. We have $aH \neq Ha$ when $a = r_{90}$, r_{270}, d, or d'.

7. Now let $K = \{\varepsilon, r_{90}, r_{180}, r_{270}\}$ be a subgroup of D_4.

(a) For each $a \in D_4$, compute the left and right cosets aK and Ka.

Solution. When $a \in D_4$ is a rotation (i.e., $a = \varepsilon$, r_{90}, r_{180}, or r_{270}), then we have $aK = Ka = K$. When $a \in D_4$ is a reflection (i.e., $a = h$, v, d, or d'), then we have $aK = Ka = \{h, v, d, d'\}$.

(b) **True or False:** $aK = Ka$ for all $a \in D_4$. **Ans for (b):** True.

Solution. True. See part (a) solution for details.

(c) **True or False:** $aK = Ka$ means $ak = ka$ for each $k \in K$.

Solution. False. With $a = h$, for example, we have $hK = Kh$ (i.e., set equality), but $h \cdot r_{90} = d$, while $r_{90} \cdot h = d'$. So, element-by-element equality does *not* hold.

8. Consider the additive group $\mathbb{Z}$ and its subgroup $H = 5\mathbb{Z}$.

(a) Compute the cosets $12 + H,\ -1 + H,\ 203 + H,\ -25 + H,$ and $101 + H$.

(b) Find all distinct cosets of H.

Solution. The distinct cosets of H are $0 + H,\ 1 + H,\ 2 + H,\ 3 + H,$ and $4 + H$. We have the following equalities of cosets:

- $12 + H = 2 + H$.
- $-1 + H = 4 + H$.
- $203 + H = 3 + H$.
- $-25 + H = 0 + H$.
- $101 + H = 1 + H$.

9. Consider the additive group $\mathbb{Z}$ and its subgroup $H = 5\mathbb{Z}$. Determine whether or not the following cosets of H are equal.

(a) $436 + H$ and $721 + H$.

Solution. We use our conjecture from Problem #5(a), namely: $a + H = b + H$ if and only if $a - b \in H$ (or $b - a \in H$). We have $436 - 721 = -285$, which is in H. Thus, the cosets $436 + H$ and $721 + H$ are equal.

(b) $-43 + H$ and $111 + H$.

Solution. We have $-43 - 111 = -154 \notin H$, so that $-43 + H \neq 111 + H$.

(c) $317 + H$ and $532 + H$.

Solution. We have $317 - 532 = -215 \in H$, so that $317 + H = 532 + H$.

10. Let H and K be subgroups of a group G. Fix $a, b \in G$ and define $aH = \{ah \mid h \in H\}$ and $bK = \{bk \mid k \in K\}$. Prove that if $aH \subseteq bK$, then $H \subseteq K$.