Abstract Algebra
Day 19 Class Work

1. Consider the (multiplicative) group U_{13} and its subgroup $H = \{1, 3, 9\}$. We just saw an example of a *coset* of H, namely $6H = \{6 \cdot 1,\ 6 \cdot 3,\ 6 \cdot 9\} = \{6, 5, 2\}$.

   (a) For each $a \in U_{13}$, compute the coset aH, i.e., compute $1H$, $2H$, $3H$, $\ldots$, $12H$.

   Suggestion: Split the work among your table members.

   (b) How many *distinct* cosets did you find?

   Ans: Less than 12.

   (c) Look ahead to Problem #5 to verify that you've found the cosets correctly.

   (d) Write down any observations you have about these cosets.

2. Consider the (additive) group $\mathbb{Z}_{12}$ and its subgroup $H = \{0, 4, 8\}$. We saw an example of a coset of H, namely $6 + H = \{6 + 0,\ 6 + 4,\ 6 + 8\} = \{6, 10, 2\}$.

   ← It's $6 + H$ instead of $6H$ for an additive group.

   (a) How many *distinct* cosets do you expect to find? How do you know?

   (b) For each $a \in \mathbb{Z}_{12}$, compute the coset $a + H$.

   ← Again, split the work among table members.

   (c) Again, look ahead to Problem #5 to check your cosets.

   (d) Write down any observations you have about these cosets.

3. Consider the (multiplicative) group D_4 and its subgroup $H = \{\varepsilon, v\}$.

   (a) How many *distinct* cosets do you expect? Explain your reasoning.

   (b) For each $a \in D_4$, compute the coset aH. The following table should help.

   Remark: For σ, $\tau \in D_4$, the "product" $\sigma \circ \tau$ is the entry in row σ and column τ. For example, the product $d \circ r_{90} = v$ is shown in bold.

$\circ$	ε	$\boldsymbol{r_{90}}$	r_{180}	r_{270}	h	v	d	d'
ε	ε	r_{90}	r_{180}	r_{270}	h	v	d	d'
r_{90}	r_{90}	r_{180}	r_{270}	ε	d'	d	h	v
r_{180}	r_{180}	r_{270}	ε	r_{90}	v	h	d'	d
r_{270}	r_{270}	ε	r_{90}	r_{180}	d	d'	v	h
h	h	d	v	d'	ε	r_{180}	r_{90}	r_{270}
v	v	d'	h	d	r_{180}	ε	r_{270}	r_{90}
$\boldsymbol{d}$	d	$\boldsymbol{v}$	d'	h	r_{270}	r_{90}	ε	r_{180}
d'	d'	h	d	v	r_{90}	r_{270}	r_{180}	ε

4. Come up with your own *multiplicative* group G and its subgroup H.

   ← Make sure G is finite.

   (a) How many distinct cosets do you expect to find?

   (b) Find all distinct cosets aH.

5. Here are some data from Problems #1 and #2, which you might find useful. ← You're welcome.

Cosets of $H = \{1, 3, 9\}$ in U_{13} are...

- $1H = 3H = 9H = \{1, 3, 9\}$
- $2H = 5H = 6H = \{2, 5, 6\}$
- $4H = 10H = 12H = \{4, 10, 12\}$
- $7H = 8H = 11H = \{7, 8, 11\}$

Cosets of $H = \{0, 4, 8\}$ in $\mathbb{Z}_{12}$ are...

- $0 + H = 4 + H = 8 + H = \{0, 4, 8\}$
- $1 + H = 5 + H = 9 + H = \{1, 5, 9\}$
- $2 + H = 6 + H = 10 + H = \{2, 6, 10\}$
- $3 + H = 7 + H = 11 + H = \{3, 7, 11\}$

Here, the cosets $3 + H$ and $11 + H$ are the same, even though $3 \neq 11$ in $\mathbb{Z}_{12}$. But could we have predicted that $3 + H = 11 + H$ *without* computing these cosets? Let's find out!

(a) Let G be a group, H a subgroup, and $a, b \in G$. Based on the examples above, describe how a and b must be be related so that:

- $a + H = b + H$ (for additive groups).

  Note: For additive groups, the relationship between a and b must be additive.

  Hint: What do you notice about $a - b$ and $b - a$?

- $aH = bH$ (for multiplicative groups). ← For example, $2H = 6H$.

  Note: Here, a and b are related multiplicatively.

(b) Prove your conjecture from part (a). ← Write a proof for the multiplicative case.

6. Let $H = \{\varepsilon, v\}$ be a subgroup of D_4. In Problem #3, you computed the *left* cosets aH.

(a) For each $a \in D_4$, compute the *right* coset $Ha = \{ha \mid h \in H\}$.

(b) **True or False:** $aH = Ha$ for all $a \in D_4$.

7. Now let $K = \{\varepsilon, r_{90}, r_{180}, r_{270}\}$ be a subgroup of D_4.

(a) For each $a \in D_4$, compute the left and right cosets aK and Ka.

(b) **True or False:** $aK = Ka$ for all $a \in D_4$. **Ans for (b):** True.

(c) **True or False:** $aK = Ka$ means $ak = ka$ for each $k \in K$.

8. Consider the additive group $\mathbb{Z}$ and its subgroup $H = 5\mathbb{Z}$.

(a) Compute the cosets $12 + H$, $-1 + H$, $203 + H$, $-25 + H$, and $101 + H$.

(b) Find all distinct cosets of H.

9. Consider the additive group $\mathbb{Z}$ and its subgroup $H = 5\mathbb{Z}$. Determine whether or not the following cosets of H are equal.

(a) $436 + H$ and $721 + H$.

(b) $-43 + H$ and $111 + H$.

(c) $317 + H$ and $532 + H$.

10. Let H and K be subgroups of a group G. Fix $a, b \in G$ and define $aH = \{ah \mid h \in H\}$ and $bK = \{bk \mid k \in K\}$. Prove that if $aH \subseteq bK$, then $H \subseteq K$.