**Abstract Algebra**
**Day 18 Class Work Solutions**

1. Recall the function $\lambda : U_{13} \to U_{13}$ where $\lambda(a) = a^3$ for all $a \in U_{13}$. The values of $\lambda(a)$ for each input $a$ in the domain $U_{13}$ are shown below: ← You're welcome.

$$\begin{array}{llll} \lambda(1)=1 & \lambda(4)=12 & \lambda(7)=5 & \lambda(10)=12 \\ \lambda(2)=8 & \lambda(5)=8 & \lambda(8)=5 & \lambda(11)=5 \\ \lambda(3)=1 & \lambda(6)=8 & \lambda(9)=1 & \lambda(12)=12 \end{array}$$

   (a) **(Review)** Show that $\lambda(a \cdot b) = \lambda(a) \cdot \lambda(b)$ for all $a, b \in U_{13}$. ← $\lambda$ is a homomorphism.

   **Solution.** We have $\lambda(a \cdot b) = (a \cdot b)^3 = a^3 \cdot b^3 = \lambda(a) \cdot \lambda(b)$.

   (b) Let $K = \{a \in U_{13} \mid \lambda(a) = 1\}$. Find the elements of $K$.

   **Solution.** $K = \{1, 3, 9\}$.

   (c) Is $K$ a subset of the domain $U_{13}$ or the codomain $U_{13}$?

   **Solution.** The *domain* $U_{13}$.

   (d) Create a multiplication table for $K$ and verify that it's a subgroup of $U_{13}$.

   **Solution.** As shown by the table below, set $K$ is closed, $1 \in K$, and every element of $K$ has an inverse in $K$ (i.e., 3 and 9 are an inverse pair, and 1 is a self inverse).

| $\cdot$ | 1 | 3 | 9 |
|---|---|---|---|
| 1 | 1 | 3 | 9 |
| 3 | 3 | 9 | 1 |
| 9 | 9 | 1 | 3 |

2. Recall the homomorphism $\varphi : \mathbb{Z} \to \mathbb{Z}_5$ where $\varphi(a) = a \pmod 5$ for all $a \in \mathbb{Z}$. ← Also recall that $\varphi(a+b) = \varphi(a) + \varphi(b)$ for all $a, b \in \mathbb{Z}$.

   (a) Let $K = \{a \in \mathbb{Z} \mid \varphi(a) = 0\}$. Find the elements of $K$.

   **Solution.** $K = 5\mathbb{Z}$, i.e., the set of all integer multiples of 5.

   (b) Is $K$ a subset of the domain or the codomain?

   **Solution.** The *domain* $\mathbb{Z}$.

   (c) Verify that $K$ is an (additive) subgroup of $\mathbb{Z}$. ← No table needed!

   Proof. Let $a, b \in 5\mathbb{Z}$ so that $a = 5k$ and $b = 5j$ where $k, j \in \mathbb{Z}$. Then $a + b = 5k + 5j = 5(k + j) \in 5\mathbb{Z}$ and so $5\mathbb{Z}$ is closed. We have $0 = 5 \cdot 0 \in 5\mathbb{Z}$. Finally, note that $-a = 5(-k) \in 5\mathbb{Z}$. Thus, $5\mathbb{Z}$ is a subgroup of $\mathbb{Z}$. ■

3. Recall the function $\gamma : \mathbb{Z}_{12} \to \mathbb{Z}_{18}$ where $\gamma(a) = 6a$ for all $a \in \mathbb{Z}_{12}$.

   (a) **(Review)** Show that $\gamma(a + b) = \gamma(a) + \gamma(b)$ for all $a, b \in \mathbb{Z}_{12}$. ← $\gamma$ is a homomorphism.

   **Solution.** We have $\gamma(a + b) = 6(a + b) = 6a + 6b = \gamma(a) + \gamma(b)$.

   (b) Let $K = \{a \in \mathbb{Z}_{12} \mid \gamma(a) = 0\}$. Find the elements of $K$. **Ans:** $K = \{0, 3, 6, 9\}$.

   **Solution.** $K = \{0, 3, 6, 9\}$.

   (c) Is $K$ a subset of the domain or the codomain?

   **Solution.** The *domain* $\mathbb{Z}_{12}$.

   (d) Create an addition table for $K$ and verify that it's a subgroup of $\mathbb{Z}_{12}$.

**Solution.** As shown by the table below, set $K$ is closed, $0 \in K$, and every element of $K$ has an inverse in $K$ (i.e., 3 and 9 are an inverse pair, and 0 and 6 are self inverses).

| + | 0 | 3 | 6 | 9 |
|---|---|---|---|---|
| 0 | 0 | 3 | 6 | 9 |
| 3 | 3 | 6 | 9 | 0 |
| 6 | 6 | 9 | 0 | 3 |
| 9 | 9 | 0 | 3 | 6 |

4. Let $\theta : G \to H$ be a group homomorphism. Let $K = \{a \in G \mid \theta(a) = \varepsilon_H\}$, where $\varepsilon_H$ is the identity element of $H$. Prove that $K$ is a subgroup of $G$. ← i.e., $G$ and $H$ are groups.

   Proof. Let $a$, $b \in K$ so that $\theta(a) = \varepsilon_H$ and $\theta(b) = \varepsilon_H$. Since $\theta$ is operation preserving, we have $\theta(ab) = \theta(a)\theta(b) = \varepsilon_H \varepsilon_H = \varepsilon_H$. Thus, $ab \in K$ and so $K$ is closed. We have $\theta(\varepsilon_G) = \varepsilon_H$, and thus $\varepsilon_G \in K$. Finally, note that $\theta(a^{-1}) = \theta(a)^{-1} = {\varepsilon_H}^{-1} = \varepsilon_H$, which shows that $a^{-1} \in K$. Therefore, $K$ is a subgroup of $G$, as desired. ■

5. Consider again the homomorphism $\lambda : U_{13} \to U_{13}$ where $\lambda(a) = a^3$ for all $a \in U_{13}$.

   (a) Let $I = \{\lambda(a) \mid a \in U_{13}\}$. Find the elements of $I$. **Ans:** $I = \{1, 5, 8, 12\}$.

   **Solution.** $I = \{1, 5, 8, 12\}$.

   (b) Is $I$ a subset of the domain $U_{13}$ or the codomain $U_{13}$?

   **Solution.** The *codomain* $U_{13}$.

   (c) Create a multiplication table for $I$ and verify that it's a subgroup of $U_{13}$.

   **Solution.** I'll leave this up to you!

6. Consider again the homomorphism $\varphi : \mathbb{Z} \to \mathbb{Z}_5$ where $\varphi(a) = a \pmod 5$ for all $a \in \mathbb{Z}$.

   (a) Let $I = \{\varphi(a) \mid a \in \mathbb{Z}\}$. Find the elements of $I$.

   **Solution.** $I = \mathbb{Z}_5$.

   (b) Is $I$ a subset of the domain or the codomain?

   **Solution.** The *codomain* $\mathbb{Z}_5$.

   (c) Anita says, "The set $I$ is all of the codomain $\mathbb{Z}_5$, because the function $\varphi$ is onto." What might she mean?

   **Solution.** The set $I$ contains all elements in the codomain that get "hit" by $\varphi$. Since $\varphi$ is onto, *every* element of the codomain $\mathbb{Z}_5$ get "hit" by the function. Thus, $I = \mathbb{Z}_5$.

7. Consider again the homomorphism $\gamma : \mathbb{Z}_{12} \to \mathbb{Z}_{18}$ where $\gamma(a) = 6a$ for all $a \in \mathbb{Z}_{12}$.

   (a) Let $I = \{\gamma(a) \mid a \in \mathbb{Z}_{12}\}$. Find the elements of $I$. ← Set $I$ has 3 elements.

   **Solution.** $I = \{0, 6, 12\}$.

   (b) Is $I$ a subset of the domain or the codomain?

   **Solution.** The *codomain* $\mathbb{Z}_{18}$.

   (c) Create an addition table for $I$ and verify that it's a subgroup of $\mathbb{Z}_{18}$.

   **Solution.** I'll leave this up to you!

8. Let $\theta : G \to H$ be a group homomorphism. Let $I = \{\theta(a) \mid a \in G\}$.

   (a) Prove that $I$ is a subgroup of $H$.

   **Hint:** For closure, start with...

   Let $i,\, j \in I$ so that $i = \theta(a)$ and $j = \theta(b)$ for some $a,\, b \in G$. ← Then show that $ij \in I$.

   (b) Suppose $I = H$. Then what can you say about the function $\theta$? Explain.

   **Solution.** The function $\theta$ is onto. See solution to Problem #6(c) for an explanation.

9. (Opening Experiment Revisited)

   (a) Use the homomorphism $\lambda$ to divide the set $U_{13}$ into 4 equal-sized subsets.

   **Hint:** One of the subsets is $K = \{1, 3, 9\}$.

   **Solution.** We have the following subsets:

$$
\begin{aligned}
K &= \{a \in U_{13} \mid \lambda(a) = 1\ \ \} = \{1,\, 3,\, 9\} \\
L &= \{a \in U_{13} \mid \lambda(a) = 8\ \ \} = \{2,\, 5,\, 6\} \\
M &= \{a \in U_{13} \mid \lambda(a) = 12\} = \{4,\, 10,\, 12\} \\
N &= \{a \in U_{13} \mid \lambda(a) = 5\ \ \} = \{7,\, 8,\, 11\}
\end{aligned}
$$

   (b) Use the homomorphism $\varphi$ to divide the set $\mathbb{Z}$ into 5 equal-sized subsets.

   (c) Use the homomorphism $\gamma$ to divide the set $\mathbb{Z}_{12}$ into 3 equal-sized subsets.

10. Let $\theta : G \to H$ be a group homomorphism. Let $K = \{a \in G \mid \theta(a) = \varepsilon_H\}$.

    **Prove:** $\theta$ is one-to-one if and only if $K = \{\varepsilon_G\}$. ← Prove both directions of "if and only if."