Abstract Algebra
Day 18 Class Work

1. Recall the function $\lambda : U_{13} \to U_{13}$ where $\lambda(a) = a^3$ for all $a \in U_{13}$. The values of $\lambda(a)$ for each input a in the domain U_{13} are shown below: ← You're welcome.

$\lambda(1) = 1$	$\lambda(4) = 12$	$\lambda(7) = 5$	$\lambda(10) = 12$
$\lambda(2) = 8$	$\lambda(5) = 8$	$\lambda(8) = 5$	$\lambda(11) = 5$
$\lambda(3) = 1$	$\lambda(6) = 8$	$\lambda(9) = 1$	$\lambda(12) = 12$

   (a) **(Review)** Show that $\lambda(a \cdot b) = \lambda(a) \cdot \lambda(b)$ for all $a, b \in U_{13}$. ← λ is a homomorphism.

   (b) Let $K = \{a \in U_{13} \mid \lambda(a) = 1\}$. Find the elements of K.

   (c) Is K a subset of the domain U_{13} or the codomain U_{13}?

   (d) Create a multiplication table for K and verify that it's a subgroup of U_{13}.

2. Recall the homomorphism $\varphi : \mathbb{Z} \to \mathbb{Z}_5$ where $\varphi(a) = a \pmod 5$ for all $a \in \mathbb{Z}$. ← Also recall that $\varphi(a + b) = \varphi(a) + \varphi(b)$ for all $a, b \in \mathbb{Z}$.

   (a) Let $K = \{a \in \mathbb{Z} \mid \varphi(a) = 0\}$. Find the elements of K.

   (b) Is K a subset of the domain or the codomain?

   (c) Verify that K is an (additive) subgroup of $\mathbb{Z}$. ← No table needed!

3. Recall the function $\gamma : \mathbb{Z}_{12} \to \mathbb{Z}_{18}$ where $\gamma(a) = 6a$ for all $a \in \mathbb{Z}_{12}$.

   (a) **(Review)** Show that $\gamma(a + b) = \gamma(a) + \gamma(b)$ for all $a, b \in \mathbb{Z}_{12}$. ← γ is a homomorphism.

   (b) Let $K = \{a \in \mathbb{Z}_{12} \mid \gamma(a) = 0\}$. Find the elements of K. **Ans:** $K = \{0, 3, 6, 9\}$.

   (c) Is K a subset of the domain or the codomain?

   (d) Create an addition table for K and verify that it's a subgroup of $\mathbb{Z}_{12}$.

4. Let $\theta : G \to H$ be a group homomorphism. Let $K = \{a \in G \mid \theta(a) = \varepsilon_H\}$, where ε_H is the identity element of H. Prove that K is a subgroup of G. ← i.e., G and H are groups.

5. Consider again the homomorphism $\lambda : U_{13} \to U_{13}$ where $\lambda(a) = a^3$ for all $a \in U_{13}$.

   (a) Let $I = \{\lambda(a) \mid a \in U_{13}\}$. Find the elements of I. **Ans:** $I = \{1, 5, 8, 12\}$.

   (b) Is I a subset of the domain U_{13} or the codomain U_{13}?

   (c) Create a multiplication table for I and verify that it's a subgroup of U_{13}.

6. Consider again the homomorphism $\varphi : \mathbb{Z} \to \mathbb{Z}_5$ where $\varphi(a) = a \pmod 5$ for all $a \in \mathbb{Z}$.

   (a) Let $I = \{\varphi(a) \mid a \in \mathbb{Z}\}$. Find the elements of I.

   (b) Is I a subset of the domain or the codomain?

   (c) Anita says, "The set I is all of the codomain $\mathbb{Z}_5$, because the function φ is onto." What might she mean?

7. Consider again the homomorphism $\gamma : \mathbb{Z}_{12} \to \mathbb{Z}_{18}$ where $\gamma(a) = 6a$ for all $a \in \mathbb{Z}_{12}$.

   (a) Let $I = \{\gamma(a) \mid a \in \mathbb{Z}_{12}\}$. Find the elements of I. ← Set I has 3 elements.

   (b) Is I a subset of the domain or the codomain?

   (c) Create an addition table for I and verify that it's a subgroup of $\mathbb{Z}_{18}$.

8. Let $\theta : G \to H$ be a group homomorphism. Let $I = \{\theta(a) \mid a \in G\}$.

   (a) Prove that I is a subgroup of H.

   Hint: For closure, start with...

   Let $i,\, j \in I$ so that $i = \theta(a)$ and $j = \theta(b)$ for some $a,\, b \in G$. ← Then show that $ij \in I$.

   (b) Suppose $I = H$. Then what can you say about the function θ? Explain.

9. (Opening Experiment Revisited)

   (a) Use the homomorphism λ to divide the set U_{13} into 4 equal-sized subsets.

   Hint: One of the subsets is $K = \{1,\, 3,\, 9\}$.

   (b) Use the homomorphism φ to divide the set $\mathbb{Z}$ into 5 equal-sized subsets.

   (c) Use the homomorphism γ to divide the set $\mathbb{Z}_{12}$ into 3 equal-sized subsets.

10. Let $\theta : G \to H$ be a group homomorphism. Let $K = \{a \in G \mid \theta(a) = \varepsilon_H\}$.

    Prove: θ is one-to-one if and only if $K = \{\varepsilon_G\}$. ← Prove both directions of "if and only if."