Abstract Algebra
Day 15 Class Work

1. It's possible that the domain and codomain of a function are the same. For instance, consider $f : U_{35} \to U_{35}$ where $f(x) = 3x$ for all $x \in U_{35}$.

   (a) Let $a, b \in U_{35}$ (domain) where $a = 8$ and $b = 22$. Compute $f(a)$ and $f(b)$. Then verify that $f(a), f(b) \in U_{35}$ (codomain) and that $f(a) \neq f(b)$. ← How do we know that a and b are in U_{35}?

   (b) Choose another pair of elements $a, b \in U_{35}$ with $a \neq b$. Then repeat part (a).

   Note: Make sure that a and b are in U_{35}, not just in $\mathbb{Z}_{35}$.

   (c) **Prove:** If $a \neq b$ in the domain, then $f(a) \neq f(b)$ in the codomain. **Hint:** Think contrapositive.

   (d) Anita wonders, "How can we be sure that $f(x)$ actually is in the codomain U_{35} and not just in $\mathbb{Z}_{35}$ for all inputs $x \in U_{35}$?" How would you respond to her? ← If $x \in U_{35}$, why must $f(x)$ also be in U_{35}?

2. Consider again $f : U_{35} \to U_{35}$ where $f(x) = 3x$ for all $x \in U_{35}$.

   (a) Let $y = 11 \in U_{35}$ (codomain). Then find $x \in U_{35}$ (domain) such that $f(x) = y$.

   (b) Repeat part (a) with another $y \in U_{35}$.

   Note: First choose the y value (in the codomain). Then find the x (in the domain).

   (c) **Prove:** Let $y \in U_{35}$. Then there exists $x \in U_{35}$ such that $f(x) = y$. ← x will depend on y.

   (d) Elizabeth wonders, "Today's Class Work problems so far remind me of the Sudoku property for U_{35}." What might she mean?

Definitions: Let $f : S \to T$ be a function from domain S to codomain T.

- We say f is *one-to-one* when it satisfies: if $a \neq b$, then $f(a) \neq f(b)$ for all $a, b \in S$.
- We say f is *onto* when for every $t \in T$, there exists $s \in S$ such that $f(s) = t$.

For example, the function that we saw in Problems #1 and #2 is both one-to-one and onto.

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3. Consider the function $\gamma : \mathbb{Z}_{12} \to \mathbb{Z}_{18}$ where $\gamma(a) = 6a$ for all $a \in \mathbb{Z}_{12}$.

   (a) Is γ one-to-one? Why or why not? **Ans to (a):** No.

   (b) Is γ onto? Why or why not?

   (c) Anita says, "I can tell right away that γ isn't onto, because its codomain is larger than its domain." What might she mean?

4. Consider the function $\varphi : \mathbb{Z} \to \mathbb{Z}_5$ where $\varphi(a) = a \pmod 5$ for all $a \in \mathbb{Z}$.

   (a) Find $\varphi(43)$ and $\varphi(-14)$. **Ans:** $\varphi(-14) = 1$.

   (b) Is φ one-to-one? Why or why not?

   (c) Is φ onto? Why or why not?

5. Let $f : \mathbb{Z} \to \mathbb{Z}$ where $f(a) = 2a$ for all $a \in \mathbb{Z}$. Explain why f is one-to-one, but not onto.

6. Describe a function $f : \mathbb{Z} \to \mathbb{Z}$ that is onto, but not one-to-one.

7. Let g be a group element with $\mathrm{ord}(g) = 18$. Define $\langle g \rangle = \{g^k \mid k \in \mathbb{Z}\}$, i.e., the set of all integer powers of g. Consider $\theta : \mathbb{Z}_{18} \to \langle g \rangle$ where $\theta(a) = g^a$ for all $a \in \mathbb{Z}_{18}$.

   Example: For $7 \in \mathbb{Z}_{18}$, we have $\theta(7) = g^7 \in \langle g \rangle$.

   (a) Let $y = g^{1001} \in \langle g \rangle$. Find $x \in \mathbb{Z}_{18}$ such that $\theta(x) = y$.

   (b) **Prove:** θ is onto.

   (c) **Prove:** θ is one-to-one. (**Hint:** Think contrapositive.)

   (d) Anita says that the sets $\mathbb{Z}_{18}$ and $\langle g \rangle$ match up completely. What might she mean?

   (e) Write down the distinct elements of $\langle g \rangle$.

8. Again, let g be a group element with $\mathrm{ord}(g) = 18$.

   (a) Compute $12 + 15$ in $\mathbb{Z}_{18}$ and $g^{12} \cdot g^{15}$ in $\langle g \rangle$.

   (b) Find the additive inverse of 12 in $\mathbb{Z}_{18}$ and the multiplicative inverse of g^{12} in $\langle g \rangle$.

   (c) Elizabeth says that $\mathbb{Z}_{18}$ and $\langle g \rangle$ behave in the same way. What might she mean?

9. **(Some Food for Thought)** Let $S = \{a, b, c, d, e\}$ and $T = \{x, y, z\}$.

   (a) How many different functions are there with domain S and codomain T? **Ans:** 243 functions.

   (b) The tables below depict two functions α and β with domain S and codomain T. Which function is onto?

s	$\alpha(s)$
a	y
b	x
c	x
d	z
e	y

s	$\beta(s)$
a	y
b	x
c	x
d	y
e	y

   (c) How many different *onto* functions are there with domain S and codomain T?