Abstract Algebra
Day 11 Class Work

1. Recall that $H = \{0, 2, 4, 6\}$ is a subgroup of $\mathbb{Z}_8$ under addition. Find *all* subgroups of $\mathbb{Z}_8$. ← How do you know that you've found them all?

2. Recall that
   - $M(\mathbb{Z}_{10})$ = the set of 2×2 matrices with entries in $\mathbb{Z}_{10}$.
   - $G(\mathbb{Z}_{10}) = \{\alpha \in M(\mathbb{Z}_{10}) \mid \alpha \text{ has a multiplicative inverse}\}$.

   (a) $M(\mathbb{Z}_{10})$ is a group under what operation?

   (b) $G(\mathbb{Z}_{10})$ is a group under what operation?

   (c) Is $G(\mathbb{Z}_{10})$ a subgroup of $M(\mathbb{Z}_{10})$? Why or why not? **Ans:** It's not. (Why not?)

3. Recall the set $S(\mathbb{Z}_{10}) = \{\alpha \in M(\mathbb{Z}_{10}) \mid \det \alpha = 1\}$.

   (a) Write down a few elements of $S(\mathbb{Z}_{10})$.

   (b) Find the multiplicative inverse of each matrix in part (a). Verify that these inverses are also in $S(\mathbb{Z}_{10})$. **Recall:** α^{-1} is given by $(\det \alpha)^{-1} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

   (c) Prove that $S(\mathbb{Z}_{10})$ is a subset of $G(\mathbb{Z}_{10})$.

   Hint: Let $\alpha \in S(\mathbb{Z}_{10})$. Why does α have a multiplicative inverse?

   (d) Prove that $S(\mathbb{Z}_{10})$ is a sub*group* of $G(\mathbb{Z}_{10})$ under multiplication. ← This should be useful: $\det(\alpha\beta) = \det \alpha \cdot \det \beta$.

   Note: You may assume that matrix multiplication is associative.

4. Find *all* subgroups of the (additive) group $\mathbb{Z}$. ← Again, how do you know you've found them all?

   Hint: The two extreme cases are $\{0\}$ and $\mathbb{Z}$ itself.

5. Given a group G, define its subset $H = \{g \in G \mid g^2 = \varepsilon\}$.

   Note: By default, assume that the operation is multiplication.

   (a) Find H if $G = U_{16}$. (Start by listing the elements of U_{16}.) **Ans:** $H = \{1, 7, 9, 15\}$.

   (b) With G and H as in part (a), construct a multiplication table for H and verify that it's a subgroup of G.

   (c) **Prove:** If G is commutative, then H is a subgroup of G.

   (d) Using a counterexample, show that the statement in part (c) is false when the group G is non-commutative.

6. Consider the subsets $H = \{0, 6\}$ and $K = \{0, 4, 8\}$ of the additive group $\mathbb{Z}_{12}$.

   (a) Verify that H is a subgroup of $\mathbb{Z}_{12}$. Do likewise for K.

   (b) With H and K as above, compute the set $H + K = \{h + k \mid h \in H, k \in K\}$. ← $H + K$ has 6 elements.

   (c) Verify that the set $H + K$ you found in part (b) is a subgroup of $\mathbb{Z}_{12}$.

   (d) **Generalize:** Let H and K be subgroups of an additive group G. Define

   $$H + K = \{h + k \mid h \in H, k \in K\}.$$

   Prove that $H + K$ is a subgroup of G.

7. **(Challenge)** Find all subgroups of $D_4 = \{\varepsilon, r_{90}, r_{180}, r_{270}, h, v, d, d'\}$. ← i.e., the group of symmetries of a square.