Abstract Algebra
Day 10 Class Work

1. **(Review)** Consider the set $U_{10} = \{1, 3, 7, 9\}$.

   (a) Complete its multiplication table below.

$\cdot$	1	3	7	9
1				
3				
7				
9				

   (b) Use the table created to check the group properties for U_{10}.

2. More generally, consider

$$U_m = \{a \in \mathbb{Z}_m \mid a \text{ has a multiplicative inverse}\}.$$

   Elizabeth claims that U_m is a group under multiplication. Help her justify each group property by answering the following questions.

   (a) Complete Elizabeth's proof below.

   Claim: U_m is closed under multiplication.

   Proof: Assume $\alpha, \beta \in U_m$.

   Then α has a multiplicative inverse k

   such that __________ in $\mathbb{Z}_m$. **Ans:** $\alpha \cdot k = 1$.

   Similarly, β has __________________

   such that __________________.

   Hence, $(\alpha\beta)\cdot(\quad) = (\quad)\cdot(\quad) = 1 \cdot 1 = 1$ in $\mathbb{Z}_m$,

   so that _____ is a multiplicative inverse of $\alpha\beta$.

   Thus, __________ as desired.

   (b) You may assume that multiplication is associative. (Move onto part (c).)

   (c) Use the definition of U_m above to explain why the multiplicative identity 1 is in U_m. ← Why is $1 \in U_m$?

   (d) Our friends are trying to prove: If $a \in U_m$, then a^{-1} is in U_m.

   - **Elizabeth:** We start by assuming that $a \in U_m$.
   - **Anita:** That means a has a multiplicative inverse. So, a^{-1} exists. We're done!
   - **Elizabeth:** Not so fast. We have to show that a^{-1} is in U_m, too.
   - **Anita:** Ok, but how do we know that a^{-1} has a multiplicative inverse? **Hint:** $a \cdot a^{-1} = 1$.

   How would you respond to Anita?

3. Let $M(\mathbb{Z}_{10})$ be the set of 2×2 matrices with entries in $\mathbb{Z}_{10}$. For example, here are two matrices in $M(\mathbb{Z}_{10})$: $\alpha = \left[\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right]$ and $\beta = \left[\begin{smallmatrix} 5 & 6 \\ 7 & 8 \end{smallmatrix}\right]$. Compute the sum $\alpha + \beta$ and product $\alpha \cdot \beta$. **Ans:** $\alpha \cdot \beta = \left[\begin{smallmatrix} 9 & 2 \\ 3 & 0 \end{smallmatrix}\right]$.

4. Recall that $M(\mathbb{Z}_{10})$ is a group under addition. Now let's consider multiplication. ← i.e., just like $\mathbb{Z}_{10}$.
   (a) Explain why $M(\mathbb{Z}_{10})$ is closed under multiplication.
   (b) You may assume that matrix multiplication is associative. (Move onto part (c).) ← Do you see why?
   (c) What is the *multiplicative* identity of $M(\mathbb{Z}_{10})$?
   (d) **True or False:** Every $\alpha \in M(\mathbb{Z}_{10})$ has a multiplicative inverse in $M(\mathbb{Z}_{10})$. **Ans:** False. (Why?)

5. Is $M(\mathbb{Z}_{10})$ a group under multiplication? Why or why not?

> Define the subset $G(\mathbb{Z}_{10}) = \{\alpha \in M(\mathbb{Z}_{10}) \mid \alpha \text{ has a multiplicative inverse}\}$.

← Analogous to U_{10}.

6. Here are some matrices in $M(\mathbb{Z}_{10})$: $\alpha = \left[\begin{smallmatrix} 1 & 2 \\ 3 & 7 \end{smallmatrix}\right]$, $\beta = \left[\begin{smallmatrix} 1 & 2 \\ 3 & 8 \end{smallmatrix}\right]$, $\gamma = \left[\begin{smallmatrix} 4 & 1 \\ 3 & 5 \end{smallmatrix}\right]$, $\delta = \left[\begin{smallmatrix} 2 & 4 \\ 1 & 5 \end{smallmatrix}\right]$.
   (a) Compute the determinant of each of these matrices.
   Recall: The *determinant* of a matrix in $M(\mathbb{Z}_{10})$ is given by $\det\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] = ad - bc$.
   (b) For each matrix, find its multiplicative inverse or explain why one doesn't exist.
   (c) Which of these are in $G(\mathbb{Z}_{10})$? **Ans:** α and γ.
   (d) Fill in the blank below:
   Theorem. Let $\alpha \in M(\mathbb{Z}_{10})$. Then $\alpha \in G(\mathbb{Z}_{10})$ if and only if ________________.

7. Explain why $G(\mathbb{Z}_{10})$ is a group under multiplication. Be sure to verify each group property.

> Define the subset $S(\mathbb{Z}_{10}) = \{\alpha \in M(\mathbb{Z}_{10}) \mid \det \alpha = 1\}$.

8. (a) Consider the matrices in Problem #6. Determine which of those are in $S(\mathbb{Z}_{10})$. **Ans:** α only.
   (b) Explain why $S(\mathbb{Z}_{10}) \subseteq G(\mathbb{Z}_{10})$.
   (c) Explain why $S(\mathbb{Z}_{10})$ is a group under multiplication. Verify each group property.

9. **(Socks-Shoes Revisited)** In $M(\mathbb{Z}_{10})$, let $\alpha = \left[\begin{smallmatrix} 3 & 4 \\ 5 & 1 \end{smallmatrix}\right]$ and $\beta = \left[\begin{smallmatrix} 7 & 1 \\ 0 & 3 \end{smallmatrix}\right]$.
   (a) Verify that α and β are in $G(\mathbb{Z}_{10})$.
   (b) Verify that $(\alpha\beta)^{-1} \neq \alpha^{-1}\beta^{-1}$, but $(\alpha\beta)^{-1} = \beta^{-1}\alpha^{-1}$.

10. **(Some Food for Thought)** U_{10} is a subset of $\mathbb{Z}_{10}$, and both are groups. Should U_{10} be called a *subgroup* of $\mathbb{Z}_{10}$? Why or why not? **Ans:** No. (Why not?)

11. **(More Food for Thought)**
    (a) Without listing them, find the number of elements in U_{35}.
    (b) Repeat part (a), but with U_{39}; with U_{55}; with U_{91}.
    (c) Repeat part (a), but with U_{pq} where p and q are distinct odd primes.