Adding matrices: Assume all entries (i.e., numbers) are in $\mathbb{Z}_{10}$.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 0 & 2 \end{bmatrix}$$

Multiplying matrices: Take the *dot product* of row and column.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1\cdot 5 + 2\cdot 7 & 1\cdot 6 + 2\cdot 8 \\ 3\cdot 5 + 4\cdot 7 & 3\cdot 6 + 4\cdot 8 \end{bmatrix} = \begin{bmatrix} 9 & 2 \\ 3 & 0 \end{bmatrix}$$

Definition: Let $M(\mathbb{Z}_{10})$ be the set of 2×2 matrices with entries in $\mathbb{Z}_{10}$.

Examples: We have α, $\beta \in M(\mathbb{Z}_{10})$, where $\alpha = \left[\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right]$ and $\beta = \left[\begin{smallmatrix} 5 & 6 \\ 7 & 8 \end{smallmatrix}\right]$.

Remarks:

- $M(\mathbb{Z}_{10})$ is closed under addition and multiplication.
- Addition and multiplication in $M(\mathbb{Z}_{10})$ are associative, i.e.,

$$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma) \quad \text{and} \quad (\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$$

for all α, β, $\gamma \in M(\mathbb{Z}_{10})$.

Claim: $M(\mathbb{Z}_{10})$ with addition is a group.

1. $M(\mathbb{Z}_{10})$ is closed under addition.

2. Addition in $M(\mathbb{Z}_{10})$ is associative.

3. $M(\mathbb{Z}_{10})$ has an *additive identity* element $\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ such that

$$\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right] + \alpha = \alpha \text{ and } \alpha + \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right] = \alpha \text{ for all } \alpha \in M(\mathbb{Z}_{10}).$$

4. Every element in $M(\mathbb{Z}_{10})$ has an additive inverse.

Example: $$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 9 & 8 \\ 7 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

But. . . $M(\mathbb{Z}_{10})$ with multiplication is *not* a group.

1. $M(\mathbb{Z}_{10})$ is closed under multiplication.

2. Multiplication in $M(\mathbb{Z}_{10})$ is associative.

3. $M(\mathbb{Z}_{10})$ has a *multiplicative identity* element $\varepsilon = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ such that

   $$\varepsilon \cdot \alpha = \alpha \text{ and } \alpha \cdot \varepsilon = \alpha \text{ for all } \alpha \in M(\mathbb{Z}_{10}).$$

4. **But. . .** not every element in $M(\mathbb{Z}_{10})$ has a multiplicative inverse.

   Example: Impossible, as the bottom row of the product is 0 0.

$$\begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Definition: Let $\alpha \in M(\mathbb{Z}_{10})$ where $\alpha = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. The *determinant of* α is given by $\det \alpha = ad - bc$. Note that $\det \alpha$ is a number in $\mathbb{Z}_{10}$.

Example: Let $\alpha = \left[\begin{smallmatrix} 2 & 1 \\ 5 & 4 \end{smallmatrix}\right]$. Then $\det \alpha = 2 \cdot 4 - 1 \cdot 5 = 3$.

Application: The multiplicative inverse of $\alpha = \left[\begin{smallmatrix} 2 & 1 \\ 5 & 4 \end{smallmatrix}\right]$ is...

$$\alpha^{-1} = 3^{-1} \cdot \begin{bmatrix} 4 & -1 \\ -5 & 2 \end{bmatrix} = 7 \cdot \begin{bmatrix} 4 & 9 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 3 \\ 5 & 4 \end{bmatrix}.$$

$\det \alpha = 3$

$3 \cdot 7 = 1$ in $\mathbb{Z}_{10}$

You should verify that $\alpha \cdot \alpha^{-1} = \varepsilon$ and $\alpha^{-1} \cdot \alpha = \varepsilon$. (**Note:** $\varepsilon = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$.)