inputs outputs

Example. Let $\sigma : \{1, 2, 3\} \to \{1, 2, 3\}$ be a function defined by

$$\sigma(1) = 3,\ \sigma(2) = 1,\ \sigma(3) = 2.$$

$$\boxed{\text{Then } \sigma \text{ is a permutation of the set } \{1, 2, 3\}, \text{ i.e., it "shuffles" the numbers 1, 2, and 3.}}$$

Non-example. The function f is *not* a permutation of $\{1, 2, 3\}$:

$$f(1) = 2,\ f(2) = 1,\ f(3) = 1.$$

Let σ and τ be permutations of $\{1, 2, 3\}$ given by

- $\sigma(1) = 3,\ \sigma(2) = 1,\ \sigma(3) = 2$
- $\tau(1) = 2,\ \tau(2) = 1,\ \tau(3) = 3$

We can compose σ and τ to obtain $\sigma \circ \tau$, which happens to be another permutation of $\{1, 2, 3\}$. Let's see...

- $(\sigma \circ \tau)(1) = \sigma(\overset{\text{inside}}{\tau(1)}) = \overset{\text{outside}}{\sigma}(\ 2\ ) = 1$
- $(\sigma \circ \tau)(2) = \sigma(\tau(2)) = \sigma(\ 1\ ) = 3$
- $(\sigma \circ \tau)(3) = \sigma(\tau(3)) = \sigma(\ 3\ ) = 2$ ☺

Note: This is just like composing symmetries.

Definition: Let S_3 be the set of all permutations of $\{1, 2, 3\}$.

Discuss in your group: Let $\sigma,\ \gamma,\ \varepsilon \in S_3$ where

- $\sigma(1) = 3,\ \sigma(2) = 1,\ \sigma(3) = 2$
- $\gamma(1) = 2,\ \gamma(2) = 3,\ \gamma(3) = 1$
- $\varepsilon(1) = 1,\ \ \varepsilon(2) = 2,\ \varepsilon(3) = 3$

Compute $\varepsilon \circ \sigma$ and $\sigma \circ \gamma$.
Notice anything?

Answer: We have $\sigma \circ \gamma = \varepsilon$, because...

- $(\sigma \circ \gamma)(1) = \sigma(\gamma(1)) = \sigma(\ 2\ ) = 1$
- $(\sigma \circ \gamma)(2) = \sigma(\gamma(2)) = \sigma(\ 3\ ) = 2$
- $(\sigma \circ \gamma)(3) = \sigma(\gamma(3)) = \sigma(\ 1\ ) = 3$

(Also, $\varepsilon \circ \sigma = \sigma$.)

Identity and Inverse

- The element $\varepsilon \in S_3$ defined by

$$\varepsilon(1) = 1, \quad \varepsilon(2) = 2, \; \varepsilon(3) = 3$$

Analogous to multiplying by 1 (or adding 0).

has the property $\varepsilon \circ \alpha = \alpha$ and $\alpha \circ \varepsilon = \alpha$ for all $\alpha \in S_3$. We call ε the identity permutation in S_3.

- We saw that $\sigma \circ \gamma = \varepsilon$ and $\gamma \circ \sigma = \varepsilon$. We say that σ and γ are inverses of each other, and we write $\gamma = \sigma^{-1}$ and $\sigma = \gamma^{-1}$.

Like $3 \cdot 5 = 1$ in U_7.

Matrix notation: Consider again $\sigma \in S_3$ defined by

$$\sigma(1) = 3,\ \sigma(2) = 1,\ \sigma(3) = 2.$$

We can write σ in *matrix form* like this:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

for S_3, top row is always 1 2 3.

bottom row contains the corresponding outputs.

Problem #2: Let α, β, $\chi \in S_n$. If $\alpha \circ \chi = \beta$, then $\chi = \alpha^{-1} \circ \beta$.

Proof: Assume $\alpha \circ \chi = \beta$.

Compose on the left by α^{-1} to get

$$\underbrace{\alpha^{-1} \circ (\alpha \circ \chi)} = \alpha^{-1} \circ \beta.$$

The left side of this equation becomes

$$\begin{aligned} \alpha^{-1} \circ (\alpha \circ \chi) &= (\alpha^{-1} \circ \alpha) \circ \chi && \text{(associative law)} \\ &= \varepsilon \circ \chi && (\alpha^{-1} \text{ is the inverse of } \alpha) \\ &= \chi && (\varepsilon \text{ is the identity}) \end{aligned}$$

Thus, $\chi = \alpha^{-1} \circ \beta$.

Scrap:

Solve $5 \cdot x = 17$.

$\downarrow * \frac{1}{5}$

$\underbrace{\frac{1}{5} \cdot (5 \cdot x)} = \frac{1}{5} \cdot 17$

$(\frac{1}{5} \cdot 5) \cdot x$

$1 \cdot x$

x

$\Rightarrow x = \frac{1}{5} \cdot 17$