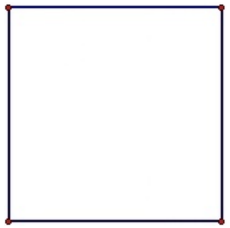
Definition. A symmetry of a square is a motion that, when applied to the square, places the square in the same space that it originally occupied.

Demonstration:



Note: $r_{90} = r_{450} = r_{810} = \cdots$

Notation: r_{90} is a counterclockwise rotation of the square (about its center) by $90°$.

$$r_{90}\left(\boxed{\begin{array}{cc}1 & 2\\ & \\ 4 & 3\end{array}}\right) = \boxed{\begin{array}{cc}2 & 3\\ & \\ 1 & 4\end{array}}$$

Key: Think of r_{90} as a *function* whose input/output is a square.

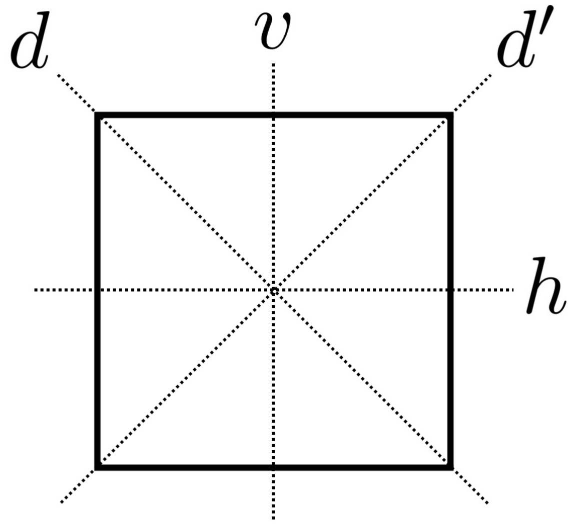
Discuss in your group: Describe all symmetries of a square.

The symmetries of a square are:

- 4 rotations: r_0, r_{90}, r_{180}, r_{270}

  Note: r_0 is often denoted ε and is called the identity.

- 4 reflections: h, v, d, d'. (These are the axis of reflection.)



Let D_4 be the set of symmetries of a square, i.e.,

$$D_4 = \{\varepsilon, r_{90}, r_{180}, r_{270}, h, v, d, d'\}.$$

- We can compose symmetries to obtain other symmetries.
- **Analogy:** We can add integers to obtain other integers.

"Composed with"

Example: $d \circ r_{90} = v$, because...

Inside function

Outside function

$$(d \circ r_{90})\left(\boxed{\begin{array}{cc}1 & 2\\ 4 & 3\end{array}}\right) = d\left(r_{90}\left(\boxed{\begin{array}{cc}1 & 2\\ 4 & 3\end{array}}\right)\right) = d\left(\boxed{\begin{array}{cc}2 & 3\\ 1 & 4\end{array}}\right) = \boxed{\begin{array}{cc}2 & 1\\ 3 & 4\end{array}}$$

Remark: This is just like $(f \circ g)(x) = f(g(x))$.

In today's Class Work, you'll complete a *composition table* for D_4:

Inside function

$r_{90} \circ d = ?$

Outside function

$\circ$	ε	r_{90}	r_{180}	r_{270}	h	v	d	d'
ε								
r_{90}			r_{270}		d'	d	h	v
r_{180}					v		d'	d
r_{270}					d	d'	v	h
h		d	v	d'		r_{180}	r_{90}	r_{270}
v		d'		d	r_{180}		r_{270}	r_{90}
d		v	d'	h	r_{270}	r_{90}		r_{180}
d'		h	d	v		r_{270}	r_{180}	

$d \circ r_{90} = v$

$\circ$	ε	r_{90}	r_{180}	r_{270}	h	v	d	d'
ε	ε	r_{90}	r_{180}	r_{270}	h	v	d	d'
r_{90}	r_{90}	r_{180}	r_{270}	ε	d'	d	h	v
r_{180}	r_{180}	r_{270}	ε	r_{90}	v	h	d'	d
r_{270}	r_{270}	ε	r_{90}	r_{180}	d	d'	v	h
h	h	d	v	d'	ε	r_{180}	r_{90}	r_{270}
v	v	d'	h	d	r_{180}	ε	r_{270}	r_{90}
d	d	v	d'	h	r_{270}	r_{90}	ε	r_{180}
d'	d'	h	d	v	r_{90}	r_{270}	r_{180}	ε

Inverse examples:

- h is a self-inverse
- $r_{90} \circ r_{270} = \varepsilon$

inverse pair

Group properties:

✓ 1. D_4 is closed.

✓ 2. Associative law, i.e., $(\sigma \circ \tau) \circ \mu = \sigma \circ (\tau \circ \mu)$. (Yes. We'll see why soon.)

✓ 3. The *identity* element ε is in D_4. $\varepsilon \circ \sigma = \sigma$, $\sigma \circ \varepsilon = \sigma$.

✓ 4. Every element in D_4 has an *inverse*.

Take $h \in D_4$, i.e., the horizontal reflection. Then...

$$C(h) = \{\sigma \in D_4 \mid \sigma \circ h = h \circ \sigma\}$$ (i.e., elements that commute with h.)

is called the *centralizer* of h in D_4.

Elements:

- $\varepsilon \in C(h)$, because $\varepsilon \circ h = h \circ \varepsilon$.
- $r_{180} \in C(h)$, because $r_{180} \circ h = h \circ r_{180}$.
- $h \in C(h)$, because $h \circ h = h \circ h$.
- $v \in C(h)$, because $v \circ h = h \circ v$.
- Also, $r_{90} \notin C(h)$, since $\underset{d'}{r_{90}} \circ h \neq h \circ \underset{d}{r_{90}}$. (Likewise for all other elements of D_4.)

$\boxed{\begin{array}{l}\textbf{Conclusion:}\\ C(h) = \{\varepsilon,\ r_{180},\ h,\ v\}\\ \text{(It's a subset of } D_4\text{.)}\end{array}}$

Table for $C(h) = \{\varepsilon,\ r_{180},\ h,\ v\}$:

$\circ$	ε	r_{180}	h	v
ε	ε	r_{180}	h	v
r_{180}	r_{180}	ε	v	h
h	h	v	ε	r_{180}
v	v	h	r_{180}	ε

Conclusion:

$C(h)$ is a *subgroup* of D_4.

Group properties:

1. $C(h)$ is closed.
2. Associative law, i.e., $(\sigma \circ \tau) \circ \mu = \sigma \circ (\tau \circ \mu)$. (Yes. We'll see why soon.)
3. $C(h)$ contains the identity ε.
4. Elements in $C(h)$ have *inverses*.