Goal: Re-prove this theorem, using a tool from *abstract algebra*.

Theorem. Let F be a field and fix $g(x) \in F[x]$.

1. If $g(x)$ is factorable, then $F[x]/\langle g(x)\rangle$ is *not* a field.
2. If $g(x)$ is unfactorable, then $F[x]/\langle g(x)\rangle$ is a field.

Discuss in your group:

- What does it mean that $5\mathbb{Z}$ is a *maximal* ideal of $\mathbb{Z}$?
- Is $12\mathbb{Z}$ maximal in $\mathbb{Z}$? No, because $12\mathbb{Z} \subsetneq 4\mathbb{Z} \subsetneq \mathbb{Z}$.

Definition. Let M be an ideal of a commutative ring R, with $M \neq R$. Then M is *maximal* in R if for any ideal A with $M \subseteq A \subseteq R$, we have $A = M$ or $A = R$.

Remark: Thus, there is no ideal A that is *strictly* between M and R.

Today's Theorem. Let M be an ideal of a commutative ring R.

$12\mathbb{Z}$ $\mathbb{Z}$ $\mathbb{Z}/12\mathbb{Z}$

1. If M is *not* maximal in R, then R/M is *not* a field. $(\mathbb{Z}/12\mathbb{Z} \cong \mathbb{Z}_{12}.)$

   (Contrapositive: If R/M is a field, then M is maximal in R.)

$5\mathbb{Z}$ $\mathbb{Z}$ $\mathbb{Z}/5\mathbb{Z}$

2. If M is maximal in R, then R/M is a field. $(\mathbb{Z}/5\mathbb{Z} \cong \mathbb{Z}_5.)$

Last time: Let F be a field and fix $g(x) \in F[x]$.

1. If $g(x)$ is factorable, then $\langle g(x) \rangle$ is *not* maximal in $F[x]$.
2. If $g(x)$ is unfactorable, then $\langle g(x) \rangle$ is maximal in $F[x]$.

$\langle g(x) \rangle$ $F[x]$

Today: Let M be an ideal of a commutative ring R.

1. If M is *not* maximal in R, then R/M is *not* a field.
2. If M is maximal in R, then R/M is a field.

Use this theorem with $R = F[x]$, $M = \langle g(x) \rangle$.

Theorem. Let F be a field and fix $g(x) \in F[x]$.

1. If $g(x)$ is factorable, then $F[x]/\langle g(x) \rangle$ is *not* a field.
2. If $g(x)$ is unfactorable, then $F[x]/\langle g(x) \rangle$ is a field.

Theorem: If M is maximal in R, then R/M is a field.

Proof outline:

- Let $a + M \neq 0 + M$ in R/M. Thus, $a \notin M$.
- Define $M + \langle a \rangle = \{m + a \cdot r \mid m \in M,\, r \in R\}$, an ideal of R.
- We have $M \subseteq M + \langle a \rangle \subseteq R$.
- Since M is maximal, $M + \langle a \rangle = M$ or $M + \langle a \rangle = R$.
- But $a = 0 + a \cdot 1 \in M + \langle a \rangle$, while $a \notin M$. Thus, $M + \langle a \rangle \neq M$.
- Then, $M + \langle a \rangle = R$, so that $1 \in M + \langle a \rangle \implies 1 = m + a \cdot r$.
- Hence, $a \cdot r + M = 1 + M$, because $1 - a \cdot r = m \in M$.
- Thus, $(a + M) \cdot (r + M) = 1 + M$, so that R/M is a field.