Recall: The quotient ring $\mathbb{Z}_7[x]/\langle x^2-1\rangle$ contains cosets of the form

$$a(x)+\langle x^2-1\rangle \quad \text{where } a(x)\in\mathbb{Z}_7[x].$$

Discuss in your group: Is $\mathbb{Z}_7[x]/\langle x^2-1\rangle$ a field? Why or why not?

Answer: No, because $\mathbb{Z}_7[x]/\langle x^2-1\rangle$ has zero divisors.

$$\begin{aligned}((x+1)+\langle x^2-1\rangle)\cdot((x-1)+\langle x^2-1\rangle) &= (x+1)(x-1)+\langle x^2-1\rangle \\ &= (x^2-1)+\langle x^2-1\rangle \\ &= 0+\langle x^2-1\rangle\end{aligned}$$

$$\neq 0+\langle x^2-1\rangle$$

Key: x^2-1 is factorable in $\mathbb{Z}_7[x]$ $\Longrightarrow$ $\mathbb{Z}_7[x]/\langle x^2-1\rangle$ is *not* a field.

Today: Consider the polynomial ring $\mathbb{R}[x]$ and a subset

$$\langle x^2+1\rangle = \{(x^2+1)\cdot q(x) \mid q(x) \in \mathbb{R}[x]\},$$

i.e., the principal ideal generated by x^2+1 (or the set of all multiples of x^2+1).

The quotient ring $\mathbb{R}[x]/\langle x^2+1\rangle$ contains cosets of the form

$$a(x) + \langle x^2+1\rangle \quad \text{where } a(x) \in \mathbb{R}[x].$$

Remark: x^2+1 is *un*factorable in $\mathbb{R}[x]$, which implies...

- Let $g(x) = x^2 + 1 \in \mathbb{R}[x]$. (**Note:** $g(x)$ is **un**factorable in $\mathbb{R}[x]$.)

- The distinct elements of $\mathbb{R}[x]/\langle g(x)\rangle$ are...

$$\mathbb{R}[x]/\langle g(x)\rangle = \{(a + bx) + \langle g(x)\rangle \mid a,\, b \in \mathbb{R}\}.$$

- We have an isomorphism $\mathbb{C} \cong \mathbb{R}[x]/\langle g(x)\rangle$ where

$$a + bi \in \mathbb{C} \text{ corresponds to } (a + bx) + \langle g(x)\rangle.$$

- **Punchline:** $\mathbb{C}$ is a field $\Longrightarrow$ $\mathbb{R}[x]/\langle g(x)\rangle$ is a field.

(Chapter 34.)

How can we generalize?