Discuss in your group: Consider $f(x) = x^2 + 1,\ g(x) = 4x + 5 \in \mathbb{R}[x]$.

(a) Compute $f(2)$ and $g(2)$, and then $f(2) + g(2)$.

(b) Compute $(f + g)(2)$, i.e., first add the polynomials, then evaluate the sum at $x = 2$.

(c) How do your answers in parts (a) and (b) compare? Is this surprising?

(d) Compare $f(2) \cdot g(2)$ and $(f \cdot g)(2)$. Any conjectures? Can you *prove* it? How?!

Remarks: $18 = 5 + 13$ $\qquad$ $65 = 5 \cdot 13$

- We have $(f + g)(2) = f(2) + g(2)$ and $(f \cdot g)(2) = f(2) \cdot g(2)$.
- These hold for all $f(x), g(x) \in \mathbb{R}[x]$. $\longleftarrow$ For you to prove!

Define $\theta : \mathbb{R}[x] \to \mathbb{R}$ where $\theta(f(x)) = f(2)$ for all $f(x) \in \mathbb{R}[x]$.

This is called the *evaluation map.*

Example: If $f(x) = x^2 + 1$, then $\theta(f(x)) = f(2) = 5$.

Observation: Using θ, we can rewrite...

$$(f+g)(2) = f(2) + g(2) \iff \theta(f(x)+g(x)) = \theta(f(x)) + \theta(g(x))$$

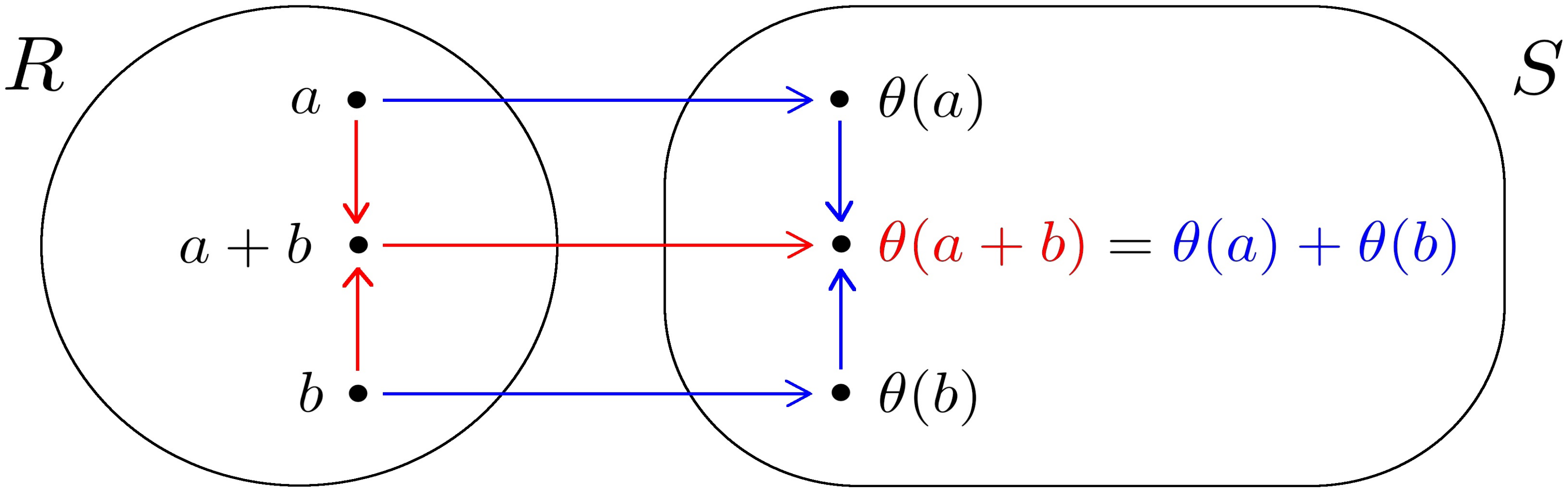
$$(f \cdot g)(2) = f(2) \cdot g(2) \iff \theta(f(x) \cdot g(x)) = \theta(f(x)) \cdot \theta(g(x))$$

Thus, θ is an example of a *ring homomorphism.* (Nothing special about 2 here.)

Definition. Given rings R and S, a function $\theta : R \to S$ is called a ring homomorphism if, for all $a, b \in R$,

$$\theta(a +_R b) = \theta(a) +_S \theta(b) \quad \text{and} \quad \theta(a \cdot_R b) = \theta(a) \cdot_S \theta(b).$$

Note: We have $R = \mathbb{R}[x]$ and $S = \mathbb{R}$ for the evaluation map example.



(This picture depicts $\theta(a + b) = \theta(a) + \theta(b)$.)

Consider $\varphi : \mathbb{Z} \to \mathbb{Z}_5$ where $\varphi(a) = a \pmod 5$ for all $a \in \mathbb{Z}$.

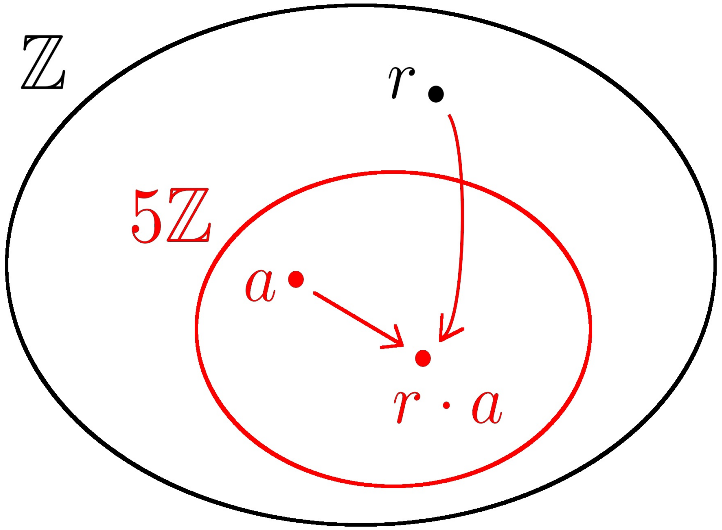
Define the *kernel* of φ as: $\ker\varphi = \{a \in \mathbb{Z} \mid \varphi(a) = 0\}$.

Note: The kernel elements map to the additive identity.

Remarks: We have $\ker\varphi = 5\mathbb{Z}$, and...

- $5\mathbb{Z}$ is an additive sub*group* of the domain $\mathbb{Z}$.
- $5\mathbb{Z}$ satisfies the product absorption property:

  If $r \in \mathbb{Z}$ (the domain) and $a \in 5\mathbb{Z}$,
  then $r \cdot a \in 5\mathbb{Z}$.



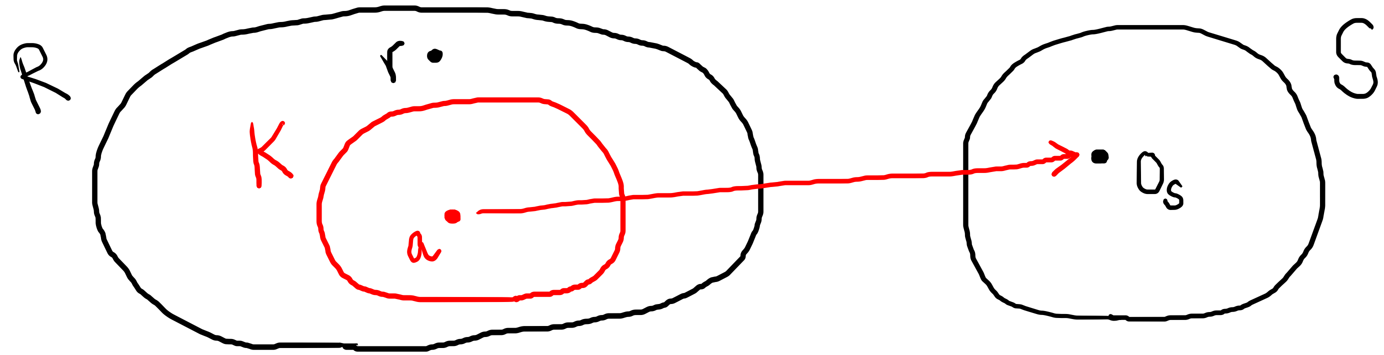
Terminology: $5\mathbb{Z}$ is an example of an *ideal* of $\mathbb{Z}$. (See Chapter 31.)

Theorem: Let $\theta : R \to S$ be a ring homomorphism with kernel

$$K = \ker\theta = \{r \in R \mid \theta(r) = 0_S\}.$$

If $r \in R$ and $a \in K$, then $r \cdot a \in K$. (Product absorption property.)

Picture:



Proof: Assume $r \in R$ and $a \in K$. Then $\theta(a) = 0_S$.

We have $\theta(r \cdot a) = \theta(r) \cdot \theta(a) = \theta(r) \cdot 0_S = 0_S$.

Hence, $\theta(r \cdot a) = 0_S$.

Thus, $r \cdot a \in K$.