Working in $F[x]$

We'll work (mostly) in $F[x]$ where F is a field.

- **Examples:** $\mathbb{R}[x]$ and $\mathbb{Z}_7[x]$, but *not* $\mathbb{Z}_9[x]$ or $\mathbb{Z}[x]$.

Big picture stuff:

The ring of integers $\mathbb{Z}$ and the polynomial ring $F[x]$ have many *structural* similarities, e.g., both have the division algorithm.

- In $\mathbb{Z}$: $a = b \cdot q + r$ where $0 \leq r < b$.

- In $F[x]$: $f(x) = g(x) \cdot q(x) + r(x)$ where $r(x)$ is "less" than $g(x)$.

Discuss in your group: Do you agree or disagree with our friends?

Elizabeth: The polynomial x^2+1 is *unfactorable*.

- x^2+1 is unfactorable in $\mathbb{R}[x]$.

$x^2 + \overset{0}{\cancel{5}}x + \overset{1}{\cancel{6}}$

- $x^2+1=(x+2)(x+3)$ is factorable in $\mathbb{Z}_5[x]$.

Key: We need to say, "x^2+1 is unfactorable in [blah]."

Anita: Actually, $x^2+1=3\cdot\left(\frac{1}{3}x^2+\frac{1}{3}\right)$ is *factorable*.

Key: Both factors must be "less" than x^2+1.

Definition. Let $f(x) \in F[x]$ where F is a field, with $\deg f(x) \geq 1$ (i.e., $f(x)$ is *not* a constant polynomial).

- We say $f(x)$ is *factorable* in $F[x]$ if we can write $f(x) = p(x) \cdot q(x)$ where $\deg p(x), \deg q(x) < \deg f(x)$.

  i.e., both factors are "less" than $f(x)$.

- Otherwise, we say $f(x)$ is *unfactorable* in $F[x]$.

Remark: Factorable / unfactorable polynomials are analogous to composite / prime integers.

$$\boxed{\deg f(x) = 1}$$

Example. $f(x) = 2x - 7 \in \mathbb{R}[x]$ is unfactorable.

Theorem. Let $f(x) \in F[x]$. If $\deg f(x) = 1$, then $f(x)$ is *unfactorable* in $F[x]$.

Remarks:

- Intuitively, we cannot factor $f(x)$ into "smaller" factors.
- See Chapter 30 reading for a complete proof.

Theorem. Let $f(x) \in F[x]$ with $\boxed{\deg f(x) \geq 2.}$

(a) If $f(x)$ has a root $\alpha \in F$, then $f(x)$ is factorable in $F[x]$.

- "α is a root of $f(x)$" means $f(\alpha) = 0$.
- Thus, $f(x) = (x - \alpha) \cdot q(x)$ where $\overset{\geq 2}{\deg f(x)} = \overset{= 1}{\deg(x - \alpha)} + \overset{< \deg f(x)}{\deg q(x)}$.
- Hence, $\deg(x - \alpha), \deg q(x) < \deg f(x)$.

(b) Assume $\deg f(x) = 2$ or 3.

If $f(x)$ has no root in F, then $f(x)$ is unfactorable in $F[x]$.

Note: We can't use Theorem (b) if $\deg f(x) > 3$. (See Problem #6.)

Theorem: Let $f(x) \in F[x]$ with $\deg f(x) = 2$ or 3.
If $f(x)$ has no root in F, then $f(x)$ is unfactorable in $F[x]$.

Proof outline: We'll prove the contrapositive, namely...

If $f(x)$ is factorable in $F[x]$, then $f(x)$ has a root in F.

Proceed similarly.

- Assume $f(x)$ is factorable in $F[x]$.
- Therefore, $f(x) = p(x) \cdot q(x)$, where $\deg p(x)$, $\deg q(x) < \deg f(x)$.
- Then $\overset{2 \text{ or } 3}{\deg f(x)} = \deg p(x) + \deg q(x) \implies \deg p(x) = 1$ or $\deg q(x) = 1$.
- If $\deg p(x) = 1$, then $p(x) = \overset{(a \neq 0)}{a}x + b$, which has a root $\alpha = -a^{-1}b$.
- Thus, $f(\alpha) = p(\alpha) \cdot q(\alpha) = 0 \cdot q(\alpha) = 0$.
- Hence, $f(x)$ has a root $\alpha \in F$.

If $p(x) = 2x + 7 \in \mathbb{R}[x]$,
then $\alpha = -7/2$
$= -2^{-1} \cdot 7$