

Discuss in your group:

- (a) Use “long division” to find the quotient and remainder when dividing 5273 by 6.

$$6 \overline{) 5273}$$

- (b) Based on your work in part (a), fill in these blanks:

$$5273 = 6 \cdot (878) + (5).$$

- (c) How do you know that you found the correct remainder?

Divide $f(x) = 5x^4 + x^3 - 3x^2 + 4x - 3$ by $g(x) = x^2 + 1$ in $\mathbb{R}[x]$.

$$\begin{array}{r}
 \\
 5x^2 + x - 8 \quad \leftarrow \text{quotient} \\
 \hline
 x^2 + 1 \quad \leftarrow \text{dividend} \\
 - (5x^4 + 5x^2) \\
 \hline
 + 8x^2 + 4x - 3 \\
 - (x^3 + x) \\
 \hline
 + 3x - 3 \\
 - (-8x^2 - 8) \\
 \hline
 + 5 \\
 + 5) \quad \leftarrow \text{remainder ("less" than the divisor)}
 \end{array}$$

divisor

Example: Let $f(x) = 5x^4 + x^3 - 3x^2 + 4x - 3$ and $g(x) = x^2 + 1$ in $\mathbb{R}[x]$.

We used **long division** to obtain: $f(x) = (x^2 + 1) \cdot q(x) + (3x + 5)$.

 quotient

Key: The degree of the *remainder* $r(x) = 3x + 5$ is less than the degree of the *divisor* $g(x) = x^2 + 1$.

Division Algorithm. Let $f(x), g(x) \in F[x]$ where F is a field with $g(x) \neq 0$.

Then there exist $q(x), r(x) \in F[x]$ such that

$$f(x) = g(x) \cdot q(x) + r(x)$$

 See Chapter 29.

with either $r(x) = 0$ or $\deg r(x) < \deg g(x)$.

Example: Let $f(x) = 4x^3 - 9x^2 + 5x - 6 \in \mathbb{R}[x]$. We have...

- $f(2) = 4 \cdot 2^3 - 9 \cdot 2^2 + 5 \cdot 2 - 6 = 0$ (Then how does $f(x)$ factor?)
- $f(x) = (x - 2) \cdot (4x^2 - x + 3)$ ← Using long division.

Terminologies: The following mean the same thing:

- $f(x)$ is a *multiple* of $x - 2$.
- $x - 2$ is a *factor* of $f(x)$. ← **Notation:** $(x - 2) \mid f(x)$.

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 - $f(x) = (x - 2) \cdot (4x^2 - x + 3)$
- harder ↙ ↘ easier

$$2 \in \mathbb{R}$$

Factor theorem. Let F be a field, $a \in F$, and $f(x) \in F[x]$. Then

$$f(a) = 0 \quad \underline{\text{if and only if}} \quad (x - a) \mid f(x).$$

Proof know-how: To prove that $g(x) \mid f(x)$, first write

$$f(x) = g(x) \cdot q(x) + r(x) \quad \longleftarrow \text{Division algorithm!}$$

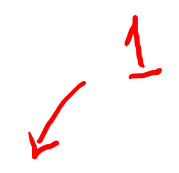
with $r(x)$ “less” than $g(x)$. Then show that $r(x) = 0$.

Theorem: Let F be a field, $a \in F$, and $f(x) \in F[x]$. If $f(a) = 0$, then $(x - a) \mid f(x)$.

Proof: Assume $f(a) = 0$.

Let $f(x) = (x - a) \cdot q(x) + r(x)$ where $q(x), r(x) \in F[x]$,

with $r(x) = 0$ or $\deg r(x) < \deg(x - a)$.



In either case, $r(x)$ is a constant. Hence, $f(x) = (x - a) \cdot q(x) + r$ where $r \in F$.

Then, $r = f(x) - (x - a) \cdot q(x)$. And substituting $x = a$ gives...

$$r = f(a) - (a - a) \cdot q(a) = 0 - 0 \cdot q(a) = 0.$$

Therefore, $f(x) = (x - a) \cdot q(x)$.

Thus, $(x - a) \mid f(x)$.