Discuss in your group:

(a) Use "long division" to find the quotient and remainder when dividing 5273 by 6.

$$6\,\overline{)\,5273}$$

(b) Based on your work in part (a), fill in these blanks:

$$5273 = 6 \cdot (\ 878\ ) + (\ 5\ ).$$

(c) How do you know that you found the correct remainder?

Divide $f(x) = 5x^4 + x^3 - 3x^2 + 4x - 3$ by $g(x) = x^2 + 1$ in $\mathbb{R}[x]$.

$$
\begin{array}{r r l}
 & & 5x^2 + x - 8 \\
 x^2 + 1 & \big) & \overline{5x^4 + x^3 - 3x^2 + 4x - 3} \\
 & & -(5x^4 \qquad\quad + 5x^2) \\
 \hline
 & & \qquad\quad x^3 - 8x^2 + 4x - 3 \\
 & & \qquad -(x^3 \qquad\quad + x) \\
 \hline
 & & \qquad\qquad\quad -8x^2 + 3x - 3 \\
 & & \qquad\qquad -(-8x^2 \qquad - 8) \\
 \hline
 & & \qquad\qquad\qquad\quad 3x + 5
\end{array}
$$

quotient

dividend

divisor

remainder ("less" than the divisor)

Example: Let $f(x) = 5x^4 + x^3 - 3x^2 + 4x - 3$ and $g(x) = x^2 + 1$ in $\mathbb{R}[x]$.

We used long division to obtain: $f(x) = (x^2 + 1) \cdot q(x) + (3x + 5)$.

quotient

Key: The degree of the *remainder* $r(x) = 3x + 5$ is less than the degree of the *divisor* $g(x) = x^2 + 1$.

Division Algorithm. Let $f(x)$, $g(x) \in F[x]$ where F is a field with $g(x) \neq 0$. Then there exist $q(x)$, $r(x) \in F[x]$ such that

$$f(x) = g(x) \cdot q(x) + r(x)$$

See Chapter 29.

with either $r(x) = 0$ or $\deg r(x) < \deg g(x)$.

Example: Let $f(x) = 4x^3 - 9x^2 + 5x - 6 \in \mathbb{R}[x]$. We have...

- $f(2) = 4 \cdot 2^3 - 9 \cdot 2^2 + 5 \cdot 2 - 6 = 0$ (Then how does $f(x)$ factor?)
- $f(x) = (x-2) \cdot (4x^2 - x + 3)$ $\leftarrow$ Using long division.

Terminologies: The following mean the same thing:

- $f(x)$ is a *multiple* of $x - 2$.
- $x - 2$ is a *factor* of $f(x)$. $\leftarrow$ **Notation:** $(x-2) \mid f(x)$.

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harder easier

- $f(x) = (x-2) \cdot (4x^2 - x + 3)$

$2 \in \mathbb{R}$

Factor theorem. Let F be a field, $a \in F$, and $f(x) \in F[x]$. Then

$$f(a) = 0 \text{ if and only if } (x-a) \mid f(x).$$

Proof know-how: To prove that $g(x) \mid f(x)$, first write

$$f(x) = g(x) \cdot q(x) + r(x) \qquad \longleftarrow \text{ Division algorithm!}$$

with $r(x)$ "less" than $g(x)$. Then show that $r(x) = 0$.

Theorem: Let F be a field, $a \in F$, and $f(x) \in F[x]$. If $f(a) = 0$, then $(x - a) \mid f(x)$.

Proof: Assume $f(a) = 0$.

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Let $f(x) = (x - a) \cdot q(x) + r(x)$ where $q(x), r(x) \in F[x]$,

$$\text{with } r(x) = 0 \text{ or } \deg r(x) < \deg(x - a).$$

In either case, $r(x)$ is a constant. Hence, $f(x) = (x - a) \cdot q(x) + r$ where $r \in F$.

Then, $r = f(x) - (x - a) \cdot q(x)$. And substituting $x = a$ gives...

$$r = f(a) - (a - a) \cdot q(a) = 0 - 0 \cdot q(a) = 0.$$

Therefore, $f(x) = (x - a) \cdot q(x)$.

Thus, $(x - a) \mid f(x)$.