**Definition.** Let $R$ be a *commutative* ring. The polynomial ring $R[x]$ is the set of all polynomials with *coefficients* in $R$.

**Examples:** $\mathbb{Z}[x]$, $\mathbb{R}[x]$, $\mathbb{Z}_5[x]$, $\mathbb{Z}_m[x]$.

- $f(x) = 2x^3 - 4x + 5$ is an element of $\mathbb{Z}[x]$. (Note that $2, -4, 5 \in \mathbb{Z}$.)

- In $\mathbb{Z}_5[x]$: $7x = 2x$, but $x^7 \neq x^2$ (i.e., reduce only the *coefficients* modulo 5).

- $x^{-1} = \dfrac{1}{x}$ is *not* a polynomial, since the exponents of $x$ must be *non-negative*.

More generally, an element of $R[x]$ has the form

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0,$$

where the *coefficients* $a_i$ are in $R$. (**Note:** $n \geq 0$.)

**Discuss in your group:** What is the *degree* of each polynomial?

$$f(x) = 2x^3 - 4x + 5 \quad \text{and} \quad g(x) = x^2 + 1.$$

**Definition:** Let $f(x)$ be a *nonzero* polynomial in $R[x]$. $\leftarrow$ $R$ is a commutative ring.
Then the degree of $f(x)$ is the highest exponent in $f(x)$.

- $\deg(2x^3 - 4x + 5) = 3$
- $\deg(x^2 + 1) = 2$
- $\deg(7) = \deg(7x^0) = 0$ $\leftarrow$ all *nonzero* constants have degree 0.

**Remarks:**

- $\deg f(x)$ must be a non-negative integer.
- The degree of $0 \in R[x]$ is undefined. (We'll see why soon.)

**Example.** Consider these polynomials in $\mathbb{Z}[x]$:

- $f(x) = 3x^{15} + 4x^3 + 2 \quad \leftarrow \deg f(x) = 15$
- $g(x) = 6x^8 + 5x + 3 \quad \leftarrow \deg g(x) = 8$

Their product is...

$$f(x) \cdot g(x) = (3x^{15}) \cdot (6x^8) + \text{(lower degree terms)}$$

$$= (3 \cdot 6) \cdot x^{15+8} + \text{(lower degree terms)}$$

$\neq 0$, since $\mathbb{Z}$ is an integral domain.

Thus, $\deg f(x) \cdot g(x) = 15 + 8$.

**Theorem.** Let $f(x),\ g(x) \in R[x]$ where $R$ is an integral domain, with $f(x),\ g(x) \neq 0$. Then

$$\deg f(x) \cdot g(x) = \deg f(x) + \deg g(x).$$

**Application:**

| | $\mathbb{Z}_7[x]$ | $\mathbb{Z}[x]$ | $\mathbb{R}[x]$ |
|---|---|---|---|
| units | 1, 2, 3, 4, 5, 6 | 1, $-1$ | All nonzero reals |

**Theorem:** If $R$ is an integral domain (like $\mathbb{Z}_7$, $\mathbb{Z}$, or $\mathbb{R}$). Then the only units in $R[x]$ are the units of $R$.

**Note:** Non-constant polynomials are *not* units in $R[x]$.

**Theorem:** Let $R$ be an integral domain. Then the only units in $R[x]$ are the units of $R$.

**Proof:** Suppose $f(x), g(x) \in R[x]$ are units (and a multiplicative inverse pair).

We must show that $f(x)$ and $g(x)$ are constant polynomials.

Since $f(x), g(x)$ are multiplicative inverses, we have $f(x) \cdot g(x) = 1$.

$$\implies \deg f(x) \cdot g(x) = \deg 1$$

$0 + 0 = 0$

$$\implies \deg f(x) + \deg g(x) = 0$$ (but degrees are non-negative, so...)

$$\implies \deg f(x) = 0 \text{ and } \deg g(x) = 0$$

Therefore, $f(x)$ and $g(x)$ are constant polynomials.

But strange things happen if $R$ is *not* an integral domain...

- In $\mathbb{Z}_9[x]$: $(1+3x)\cdot(1+6x) = 1 + 9x + 18x^2 = 1$, i.e., non-constant units!
- In $\mathbb{Z}_6[x]$: $(2+4x)\cdot 3x^2 = 6x^2 + 12x^3 = 0$, i.e., zero divisors!

**Punchline:**

- We want to work in $R[x]$ where $R$ is an integral domain.
- In fact, $R$ being a field is even better (e.g., $\mathbb{Z}_7[x]$, $\mathbb{R}[x]$), and that's what we'll do most of the time.