Let G be a (multiplicative) group and H its subgroup.

Coset multiplication shortcut: For $aH, bH \in G/H$, $aH \cdot bH = (ab)H$.

We know: ??? $\Longrightarrow$ CM shortcut holds in G/H $\Longrightarrow$ G/H is a group

Today's goal: Complete the missing piece ???.

Coset multiplication shortcut: For $aH, bH \in G/H$, $\boxed{aH \cdot bH = (ab)H.}$

Digging deeper: Both $aH \cdot bH$ and $(ab)H$ are *sets*.

- An element in $aH \cdot bH$ looks like $\alpha\beta$, where $\alpha \in aH$ and $\beta \in bH$.
- An element in $(ab)H$ looks like $(ab)h$ for some $h \in H$.

Set equality: To prove that $aH \cdot bH = (ab)H$, we must show...

- $(ab)H \subseteq aH \cdot bH$ (this is *always* true)
- $aH \cdot bH \subseteq (ab)H$ (this is *sometimes* true... but when?)

Problem #4: Let $K = \{\varepsilon, r_{90}, r_{180}, r_{270}\}$ be a subgroup of D_4.

- $vK = \{v\varepsilon,\ vr_{90},\ vr_{180},\ vr_{270}\} = \{v,\ d',\ h,\ d\}$ $\leftarrow$ left coset
- $Kv = \{\varepsilon v,\ r_{90}v,\ r_{180}v,\ r_{270}v\} = \{v,\ d,\ h,\ d'\}$ $\leftarrow$ right coset

Observations:

- $vK = Kv$. (In fact, we have $aK = Ka$ for all $a \in D_4$.)
- But that does *not* mean $vk = kv$ for each $k \in K$.
- **Key:** Instead, we must say $vk = jv$ for some $j \in K$.

$\rightarrow$ $vr_{90} = r_{270}v$, for example.

Definition. Let H be a subgroup of a group G. Then H is called a *normal subgroup* of G if $gH = Hg$ for all $g \in G$.

- In other words, all left and right cosets of H are equal.
- We often say, "H is normal in G."

Example. $K = \{\varepsilon, r_{90}, r_{180}, r_{270}\}$ is a normal subgroup of D_4, because $aK = Ka$ for all $a \in D_4$.

Question: Why should we care about normal subgroups?

Answer: H is a normal subgroup $\Longrightarrow$ CM shortcut holds in G/H $\Longrightarrow$ G/H is a group

Theorem. If H is normal in G, then $aH \cdot bH = (ab)H$.

- We must show $aH \cdot bH \subseteq (ab)H$ and ✓ $(ab)H \subseteq aH \cdot bH$.
- But $(ab)H \subseteq aH \cdot bH$ is *always* true. (Proved in Day 21.)

Proof of $aH \cdot bH \subseteq (ab)H$: Assume H is normal in G.

Let $x \in aH \cdot bH$ so that $x = ah \cdot bk$ for some $h, k \in H$.

Since H is a normal subgroup, we have $Hb = bH$.

Hence, $hb = bj$ for some $j \in H$.

Therefore, $x = ah \cdot bk = a(hb)k = a(bj)k = (ab)(jk) \in (ab)H$.

Thus, $x \in (ab)H$ so that $aH \cdot bH \subseteq (ab)H$.

Important:

Read about the

“Normal subgroup test”

in Chapter 24.