Let G be a (multiplicative) group and H a subgroup.

Coset multiplication shortcut: For $aH, bH \in G/H$, $\boxed{aH \cdot bH = (ab)H.}$

IF the shortcut holds in G/H, then…

G/H is a quotient group under coset multiplication.

We know: ??? $\Longrightarrow$ CM shortcut holds in G/H $\Longrightarrow$ G/H is a group

- We'll explore ??? next time.
- For today, assume that the shortcut holds.

Let G be a group and H a subgroup. Suppose that G/H satisfies the coset multiplication shortcut.

Discuss in your group: Explain why

Last time: $(aH)^{-1} = a^{-1}H$.

$$(aH)^3 = a^3H \text{ and } (aH)^{-3} = a^{-3}H \quad \text{for all } aH \in G/H.$$

- $(aH)^3 = aH \cdot aH \cdot aH \overset{CM}{=} a^3H.$
- $(aH)^{-3} = ((aH)^{-1})^3 = (aH)^{-1} \cdot (aH)^{-1} \cdot (aH)^{-1}$
  $= a^{-1}H \cdot a^{-1}H \cdot a^{-1}H \overset{CM}{=} (a^{-1})^3H = a^{-3}H.$

Conclusion: $(aH)^k = a^kH$ for all $k \in \mathbb{Z}$.

Theorem: Suppose $[G : H] = n$. Then $g^n \in H$ for all $g \in G$.

Proof: Let $g \in G$. We must show that $g^n \in H$.

We know that $n = [G : H] =$ size of G/H.

Consider $gH \in G/H$, and let $d = \text{ord}(gH)$ in G/H.

Then $d \mid n$ so that $n = d \cdot k$ for some integer k.

Thus, $(gH)^n = (gH)^{dk} = ((gH)^d)^k = (\varepsilon H)^k = \varepsilon H$,

so that $(gH)^n = \varepsilon H$.

Also, $(gH)^n = g^n H$, and so $g^n H = \varepsilon H = H$.

Therefore, $g^n \in H$.

Scrap: Let $g \in G$.

* Work with $gH \in G/H$

* $n = [G:H]$
$= \text{size of } G/H$

$\Longrightarrow (gH)^n = \varepsilon H$

$\Longrightarrow g^n H = H$

$\Longrightarrow g^n \in H$.