Last time: Consider the subgroup $H = \{1, 3, 9\}$ of U_{13}.

- The set of distinct cosets of H is: $U_{13}/H = \{1H,\ 2H,\ 4H,\ 7H\}$.

- The set U_{13}/H is a quotient group under coset multiplication.

  For example...

$$
\begin{aligned}
2H \cdot 7H &= \{2,\ 5,\ 6\} \cdot \{7,\ 8,\ 11\} \\
&= \{2\cdot 7,\ 2\cdot 8,\ 2\cdot 11,\ 5\cdot 7,\ 5\cdot 8,\ 5\cdot 11,\ 6\cdot 7,\ 6\cdot 8,\ 6\cdot 11\} \\
&= \{1, 3, 9, 9, 1, 3, 3, 9, 1\} \\
&= 1H
\end{aligned}
$$

Group table for U_{13}/H

$\cdot$	$1H$	$2H$	$4H$	$7H$
$1H$	$1H$	$2H$	$4H$	$7H$
$2H$	$2H$	$4H$	$7H$	$1H$
$4H$	$4H$	$7H$	$1H$	$2H$
$7H$	$7H$	$1H$	$2H$	$4H$

Group properties:

- Closure.
- Associativity. (Chapter 21 reading.)
- $1H$ is the identity element of U_{13}/H.
- **Inverses:**
    - $7H = (2H)^{-1}$ and $2H = (7H)^{-1}$.
    - Also, $1H$ and $4H$ are self inverses.

Coset multiplication shortcut: For $aH, bH \in U_{13}/H$,

$$\boxed{aH \cdot bH = (ab)H.}$$

For example, we *should* have $2H \cdot 7H = 14H$, and we've seen indeed that $2H \cdot 7H = 1H$. (Note that $14H = 1H$.)

What we know so far:

ex $G = U_{13}$

- We proved: G is commutative $\Longrightarrow$ Shortcut holds in G/H

- But we saw that the shortcut also holds in D_4/Z, where $Z = \{\varepsilon, r_{180}\}$, even though D_4 is non-commutative.

Consider the subgroup $H = \langle 10 \rangle$ of the multiplicative group U_{37}.

Problem #4: Find the inverse of $15H$ in U_{37}/H.

- In U_{37}: $15 \cdot 5 = 1 \implies 15^{-1} = 5.$
- In U_{37}/H: $15H \cdot 5H = (15 \cdot 5)H = 1H \implies (15H)^{-1} = 5H = 15^{-1}H$

Conclusion: $(15H)^{-1} = 15^{-1}H.$

Generalize: $\boxed{(aH)^{-1} = a^{-1}H.}$

Verify: U_{37}/H **is a quotient group**

$$\boxed{aH \cdot bH = (ab)H}$$

Since U_{37}/H satisfies the coset multiplication shortcut...

1. U_{37}/H is closed under coset multiplication, i.e.,

$$aH \cdot bH = (ab)H \in U_{37}/H \text{ for all } aH, bH \in U_{37}/H.$$

2. Coset multiplication is associative. (See Chapter 21.)

3. U_{37}/H contains an identity element $1H$ such that

$$1H \cdot aH = (1 \cdot a)H = aH \text{ for all } aH \in U_{37}/H. \quad (\text{Also, } aH \cdot 1H = aH.)$$

4. Each $aH \in U_{37}/H$ has an inverse $a^{-1}H \in U_{37}/H$ where

$$aH \cdot a^{-1}H = (a \cdot a^{-1})H = 1H. \quad (\text{Also, } a^{-1}H \cdot aH = 1H.)$$

What we know so far (again)

- So we just proved:

$$\boxed{\text{Shortcut holds in } G/H} \implies \boxed{G/H \text{ is a quotient group}}$$

- **Question:** When does the shortcut hold?

  (When G is commutative, yes. But there's more to the story... Stay tuned!)