Opening experiment

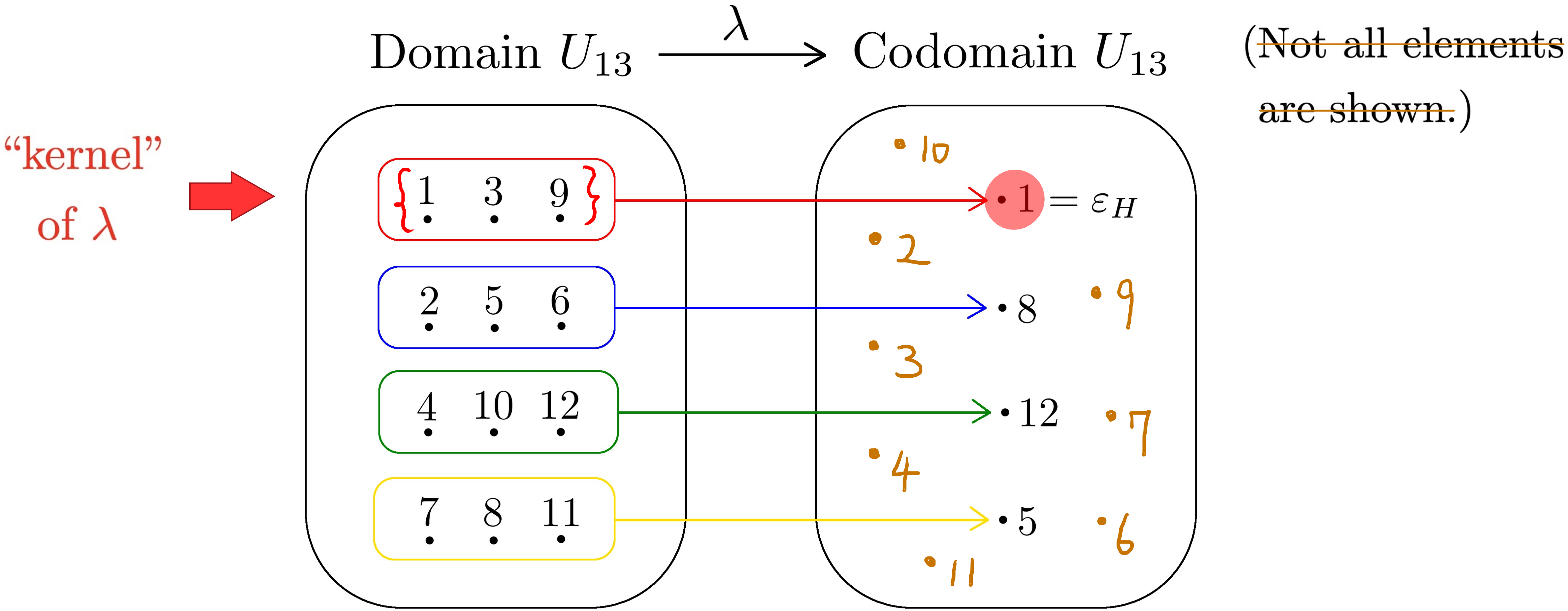
Discuss in your group: Divide the set

$$U_{13} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$

into 4 equal-sized subsets, with 3 elements each.

(Yes, there is a correct answer.)

Example. $\lambda : U_{13} \to U_{13}$ where $\lambda(a) = a^3$ for all $a \in U_{13}$.



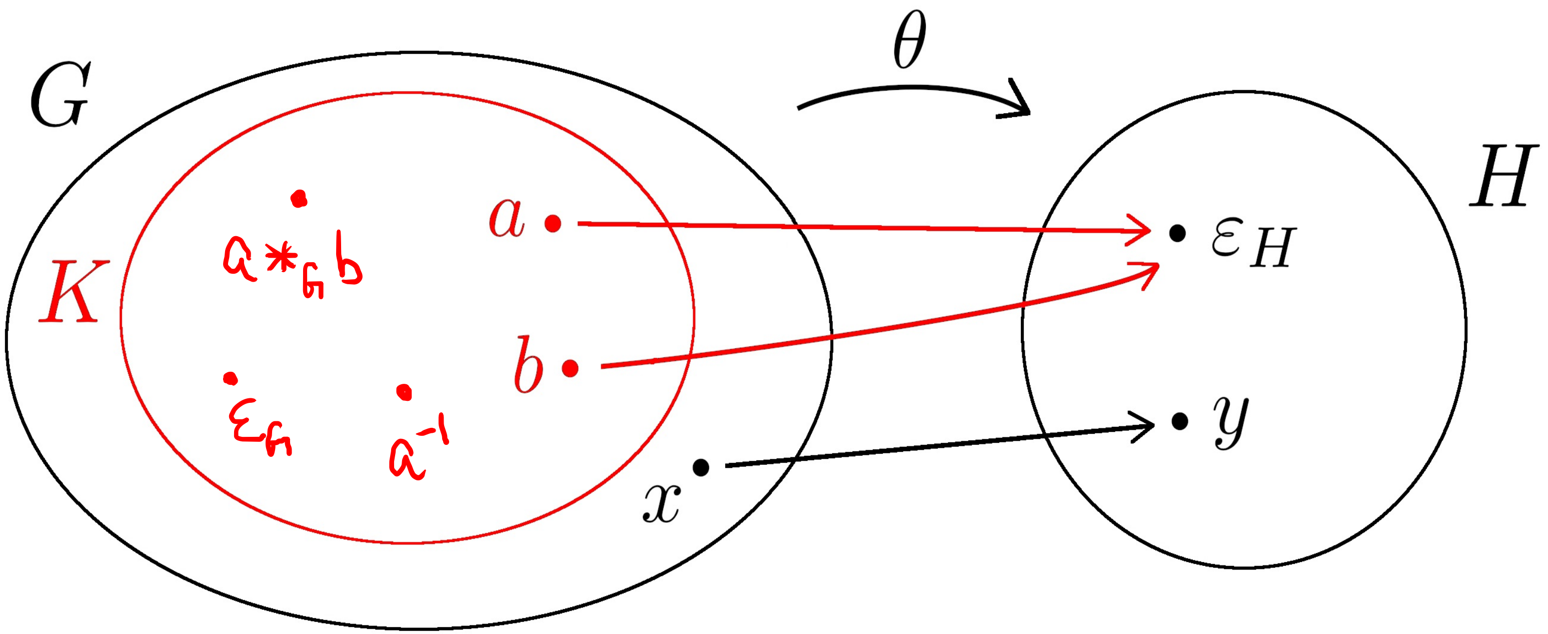
Observation: A group homomorphism *partitions* the domain into equal-sized subsets. (Why? We'll see soon!)

Kernel of a homomorphism

Let $\theta : G \to H$ be a group homomorphism. Define

$$K = \{a \in G \mid \theta(a) = \varepsilon_H\} \subseteq G. \qquad (\varepsilon_H = \text{identity of } H.)$$

Then K is called the kernel of θ and is denoted $\ker\theta$.



Theorem:

K is a subgroup of G.

Theorem. Let $\theta : G \to H$ be a group homomorphism. Then $K = \ker\theta$ is a subgroup of G.

$a \in K$ means $\theta(a) = \varepsilon_H$.

Proof: Let $a, b \in K$. Then $\theta(a) = \varepsilon_H$, $\theta(b) = \varepsilon_H$.

Since θ is operation preserving,

$$\theta(a *_G b) = \theta(a) *_H \theta(b) = \varepsilon_H *_H \varepsilon_H = \varepsilon_H.$$

Thus, $a *_G b \in K$ so that K is closed.

Since θ preserves the identity, we have $\theta(\varepsilon_G) = \varepsilon_H$.

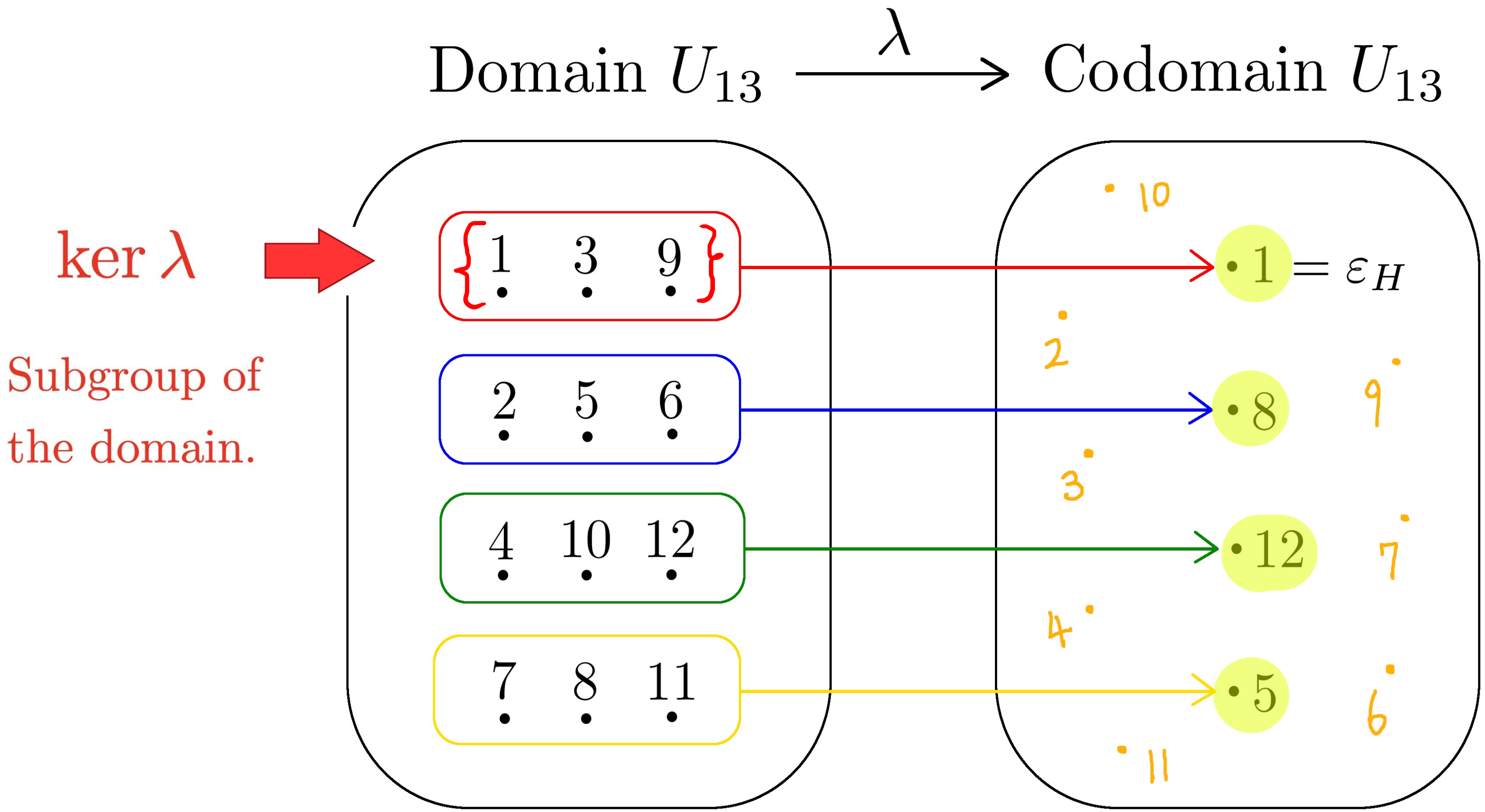
And thus, $\varepsilon_G \in K$.

Finally, $\theta(a^{-1}) = \theta(a)^{-1} = \varepsilon_H^{-1} = \varepsilon_H$. Thus, $a^{-1} \in K$.

Hence K is a subgroup of G.

Image of a homomorphism

Example. $\lambda : U_{13} \to U_{13}$ where $\lambda(a) = a^3$ for all $a \in U_{13}$.



Subgroup of the codomain.

"image" of λ

$= \{1, 8, 12, 5\}$

$\neq U_{13}$

λ is *not* onto.