**$\mathbb{Z}_{10}$ revisited:** Consider $\mathbb{Z}_{10} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ again.

Recall that $\mathbb{Z}_{10}$ is a group under addition.

**Newsflash:** We can add *and* multiply in $\mathbb{Z}_{10}$.

**Discuss in your group:** Is $\mathbb{Z}_{10}$ a group under multiplication? No.

**Group properties:**

✓ 1. Closure: If $a, b \in \mathbb{Z}_{10}$, then $a \cdot b \in \mathbb{Z}_{10}$.

✓ 2. Associativity: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

✓ 3. (Multiplicative) Identity: $1 \in \mathbb{Z}_{10}$.

☹ 4. (Multiplicative) Inverses: $0 \cdot x = 1$ Not possible.

**Dilemma:** $\mathbb{Z}_{10}$ is *not* a group under multiplication, because not every element has a multiplicative inverse.

**Example:**

- $4 \cdot x = 1$ is *not* possible in $\mathbb{Z}_{10}$, so $4^{-1}$ does *not* exist.
- (But $4 + x = 0$ is possible in $\mathbb{Z}_{10}$, so $-4 = 6$ does exist.)

**Question:** How did we salvage the situation?

**Answer:** Define this subset...

$$U_{10} = \{a \in \mathbb{Z}_{10} \mid a \text{ has a multiplicative inverse}\}$$
$$= \{1, 3, 7, 9\}$$

**Matrix group:** Let $M(\mathbb{Z}_{10})$ be the set of $2 \times 2$ matrices with entries in $\mathbb{Z}_{10}$.

Recall that $M(\mathbb{Z}_{10})$ is a group under addition.

**Newsflash:** We can add *and* multiply in $M(\mathbb{Z}_{10})$.

**Question:** Is $M(\mathbb{Z}_{10})$ a group under multiplication? No.

**Group properties:**

✓ 1. Closure: If $\alpha, \beta \in M(\mathbb{Z}_{10})$, then $\alpha \cdot \beta \in M(\mathbb{Z}_{10})$.

✓ 2. Associativity: $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$.

✓ 3. (Multiplicative) Identity: $\varepsilon = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in M(\mathbb{Z}_{10})$.

4. (Multiplicative) Inverses:

This isn't possible:

$$\begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

So, $\begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix}^{-1}$ does *not* exist.

**Definition:** $G(\mathbb{Z}_{10}) = \{\alpha \in M(\mathbb{Z}_{10}) \mid \alpha \text{ has a multiplicative inverse}\}$.

**Theorem:** $G(\mathbb{Z}_{10})$ is a multiplicative group.

✓1. $G(\mathbb{Z}_{10})$ is closed under multiplication. (Proof is similar to $U_m$.)

✓2. Matrix multiplication is associative.

✓3. $\varepsilon = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ is in $G(\mathbb{Z}_{10})$, because $\varepsilon \cdot \varepsilon = \varepsilon$ (i.e., $\varepsilon$ is a self inverse).

✓4. $\alpha \in G(\mathbb{Z}_{10}) \implies \alpha^{-1}$ exists such that $\alpha \cdot \alpha^{-1} = \varepsilon$.

$\implies \alpha^{-1}$ has a multiplicative inverse, namely $\alpha$.

$\implies \alpha^{-1} \in G(\mathbb{Z}_{10})$.

## Similarity between $U_{10}$ and $G(\mathbb{Z}_{10})$

Both multiplicative groups $U_{10}$ and $G(\mathbb{Z}_{10})$ have a "trick" that allows us to easily determine whether or not an element is in the group.

**Examples:**

- $8 \in U_{35}$, because $\gcd(8, 35) = 1$.
- $10 \notin U_{35}$, since $\gcd(10, 35) \neq 1$.
- $\alpha = \left[\begin{smallmatrix} 2 & 1 \\ 5 & 4 \end{smallmatrix}\right] \in G(\mathbb{Z}_{10})$, as $\det \alpha = 3 \in U_{10}$.
- $\beta = \left[\begin{smallmatrix} 2 & 1 \\ 3 & 4 \end{smallmatrix}\right] \notin G(\mathbb{Z}_{10})$, since $\det \beta = 5 \notin U_{10}$.