Problems from the Putnam

1967 (A1) Let \( f(x) = a_1 \sin x + a_2 \sin 2x + \cdots + a_n \sin nx \) where \( a_1, a_2, \ldots, a_n \) are real numbers and where \( n \) is a positive integer. Given that \( |f(x)| \leq |\sin x| \) for all real \( x \) prove that
\[
|a_1 + 2a_2 + \cdots + na_n| \leq 1.
\]

1971 (A2) Find all polynomials \( P(x) \) such that \( P(x^2 + 1) = (P(x))^2 + 1 \) and \( P(0) = 0 \).

1971 (B2) Let \( f(x) \) be a real valued function defined for all real \( x \) except for \( x = 0 \) and \( x = 1 \) and satisfying the functional equation:
\[
f(x) + f\left(\frac{x-1}{x}\right) = 1 + x.
\]
Find all functions \( f(x) \) satisfying these conditions.

1978 (A1) Let \( A \) be any set of 20 distinct integers from the arithmetic progression 1, 4, 7, ..., 100. Prove that there must be two distinct integers in \( A \) whose sum is 104.

1951 (B3) Show that if \( x \) is positive, then \( \log_e (1 + 1/x) > 1/(1 + x) \).

1967 (A2) Find all continuous positive functions \( f(x) \), for \( 0 \leq x \leq 1 \), such that
\[
\int_0^1 f(x) \, dx = 1
\]
\[
\int_0^1 f(x)x \, dx = \alpha
\]
\[
\int_0^1 f(x)x^2 \, dx = \alpha^2
\]
where \( \alpha \) is a given real number.

1998 (A1) A right circular cone has a base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained on the base of the cone. What is the side length of the cube?

1994 (A4) Let \( G \) be a group with identity \( e \) and \( \phi : G \to G \) a function such that
\[
\phi(g_1)\phi(g_2)\phi(g_3) = \phi(h_1)\phi(h_2)\phi(h_3)
\]
whenever \( g_1g_2g_3 = e = h_1h_2h_3 \). Prove that there exists an element \( a \in G \) such that \( \psi(x) = a\phi(x) \) is a homomorphism (that is, \( \psi(xy) = \psi(x)\psi(y) \) for all \( x, y \in G \)).