ELEVENTH ANNUAL
NORTH CENTRAL SECTION MAA
TEAM CONTEST

November 10, 2007, 9:00 a.m. to 12:00 noon

NO BOOKS, NOTES, CALCULATORS, COMPUTERS OR NON-TEAM-MEMBERS may be consulted.

PLEASE BEGIN EACH PROBLEM ON A NEW SHEET OF PAPER. Team identification and problem number should be clearly given at the top of each sheet of paper submitted.

Each problem counts 10 points. Partial credit for significant but incomplete work. For full credit, answers must be fully justified. But in some cases this may simply mean showing all work and reasoning. Have fun!

1. Logarithmic sum.

The function $f$ is defined on the positive integers by

$$f(n) = \begin{cases} 
\log_{32} n & \text{if } \log_{32} n \text{ is rational} \\
1 & \text{otherwise.}
\end{cases}$$

Evaluate

$$\sum_{n=1}^{2007} f(n).$$

2. A magic square.

In a magic square, all row sums, column sums, and the two diagonal sums are equal. Three entries are given in a $3 \times 3$ square at the right. Fill in the remaining entries with any real numbers whatever to make it a magic square.

3. The coefficient of $x^2$.

Let $P_0(x) = x^3 + ax^2 - 1018x + 2007$, and for integers $n \geq 1$, let $P_n(x) = P_{n-1}(x - n)$. If the coefficient of $x$ in $P_{10}(x)$ is 2007, what is the coefficient of $x^2$ in $P_{10}(x)$?
4. Integral of fractional part.

For real $x$, let $\langle x \rangle$ denote the fractional part of $x$. Thus, $\langle x \rangle = x - \lfloor x \rfloor$, where $\lfloor x \rfloor$ is the greatest integer less than or equal to $x$. Evaluate

$$\int_{-1}^{1} (x^2 + 2x - 3) \, dx.$$ 

5. Difference of squares.

Find all ordered pairs $(a, b)$ of positive integers such that

$$a^2 - b^2 = 2007.$$ 


Find all pairs of positive integers $(x, y)$ with $x \leq y$ such that $\sqrt{x} + \sqrt{y} = \sqrt{2007}$.

7. Difference of square roots.

Is there an integer $N$ satisfying the following equation?

$$\left(\sqrt{2007} - \sqrt{2006}\right)^{2008} = \sqrt{N} - \sqrt{N - 1}.$$ 

8. Group product.

Elements $a_1, a_2, \ldots, a_n$ are selected sequentially at random, with replacement (so they are not necessarily distinct) from a multiplicative group $G$ of order $n$, with identity 1. Show that there exist integers $r$ and $s$, with $1 \leq r \leq s \leq n$, such that

$$\prod_{k=r}^{s} a_k = 1.$$ 

9. Limit of a sequence of integrals.

Evaluate

$$\lim_{n \to \infty} \int_{n}^{2n} \frac{n^3 x \, dx}{x^3 + 1}.$$ 

10. Lattice points on a curve.

Show that the only lattice points $(x, y)$ on the curve

$$y^2 = x^4 + x^3 + x^2 + x + 1$$

are $(-1, \pm 1), (0, \pm 1)$ and $(3, \pm 11)$. (Lattice points are points with integer coordinates.)
Each problem number is followed by an 11-tuple \( (a_{10}, a_9, a_8, a_7, a_6, a_5, a_4, a_3, a_2, a_1, a_0) \), where \( a_k \) is the number of teams that scored \( k \) points on the problem.

1. Logarithmic sum. \((30,1,5,0,0,0,0,6,4,1,6)\)

The sum is \( \sum_{n=1}^{2007} f(n) = 2007 \). Note that \( \log_{32} n \) is rational if and only if \( n \) is an integral power of two, and \( \log_{32} 2^r = r/5 \). Now \( 1 \leq 2^r \leq 2007 \) if and only if \( 0 \leq r \leq 10 \), and the sum of \( f(n) \) over these eleven values of \( n \) is

\[
\frac{0}{5} + \frac{1}{5} + \frac{2}{5} + \cdots + \frac{10}{5} = \frac{55}{5} = 11.
\]

The sum of \( f(n) \) over the remaining \((2007 - 11) \) values of \( n \) is 2007 - 11, so

\[
\sum_{n=1}^{2007} f(n) = 2007.
\]

2. A magic square. \((46,0,0,1,0,0,0,4,1,0,1)\)

The unique solution is shown at the right. Let \( x \) be the number in the upper left corner. Then the common sum is \( x + 27 \), and the lower left entry must be 11. To make the lower left to upper right diagonal sum to \( x + 27 \), the central entry must be \( x + 9 \). Then to make the sum on row 2 be \( x + 27 \), the center right element must be 2. To make column 2 sum to \( x + 27 \), the bottom center element must be \(-2 \). Now either the third row or third column tells us that the lower right corner must contain \( x + 18 \), and then the sum on the remaining diagonal is \( 3x + 27 = x + 27 \), so \( x = 0 \).

3. The coefficient of \( x^2 \). \((23,0,6,3,2,4,1,0,1,0,13)\)

If the coefficient of \( x \) in \( P_{10}(x) \) is 2007, then the coefficient of \( x^2 \) in \( P_{10}(x) \) is \(-110 \). We first show that \( a = 55 \). We have \( P_1(x) = P_0(x - 1) \), \( P_2(x) = P_1(x - 2) = P_0(x - 3) \), and by an easy induction, in general, \( P_n(x) = P_0(x - n(n + 1)/2) \). In particular,

\[
P_{10}(x) = P_0(x - 55) = (x - 55)^2 + a(x - 55)^2 - 1018(x - 55) + 2007,
\]

and the coefficient of \( x \) is \( 3 \cdot 55^2 - 110a - 1018 \). For this to equal 2007 we need \( 110a = 3 \cdot 55^2 - 1018 - 2007 = 6050 \), making \( a = 55 \). Then the coefficient of \( x^2 \) in \( P_{10}(x) \) is \(-3 \cdot 55 + a = -110 \).
4. Integral of fractional part. \((17,1,0,7,1,4,1,4,0,1,17)\)

The value is \(\frac{\sqrt{2} + \sqrt{3} - 7}{3}\). Note that \(\langle x + n \rangle = \langle x \rangle\) for every real \(x\) and every integer \(n\), so \(\langle x^2 + 2x - 3 \rangle = \langle (x + 1)^2 \rangle\). Using the substitution \(u = x + 1\), then, we have

\[
\int_{-1}^{1} \langle x^2 + 2x - 3 \rangle \, dx = \int_{-1}^{1} \langle (x + 1)^2 \rangle \, dx = \int_{0}^{2} \langle u^2 \rangle \, du.
\]

For \(0 \leq u < 1\), we have \(0 \leq u^2 < 1\) and \(\langle u^2 \rangle = u^2\). For \(1 \leq u < \sqrt{2}\), we have \(1 \leq u^2 < 2\), and \(\langle u^2 \rangle = u^2 - 1\). For \(\sqrt{2} \leq u < \sqrt{3}\), \(\langle u^2 \rangle = u^2 - 2\), and for \(\sqrt{3} \leq u < 2\), \(\langle u^2 \rangle = u^2 - 3\). Thus

\[
\int_{0}^{1} \langle u^2 \rangle \, du = \int_{0}^{1} u^2 \, du + \int_{1}^{\sqrt{2}} (u^2 - 1) \, du + \int_{\sqrt{2}}^{\sqrt{3}} (u^2 - 2) \, du + \int_{\sqrt{3}}^{2} (u^2 - 3) \, du
\]

\[
= \int_{0}^{2} u^2 \, du - 1(\sqrt{2} - 1) - 2(\sqrt{3} - \sqrt{2}) - 3(2 - \sqrt{3})
\]

\[
= \frac{8}{3} + 1 - 6 + \sqrt{2} + \sqrt{3}
\]

\[
= \sqrt{2} + \sqrt{3} - \frac{7}{3}.
\]

5. Difference of squares. \((26,1,0,6,0,1,0,6,1,0,12)\)

There are three such pairs, namely \([(1004, 1003), (336, 333)\) and \((116, 107)\)]. If \(2007 = a^2 - b^2\), then

\]

The factorization \((a + b)(a - b) = (2007)(1)\) gives \(a = 1004, b = 1003\). The other two factorizations yield \((a, b) = (336, 333)\) and \((116, 107)\), respectively.

6. Sum of square roots. \((13,0,1,10,0,1,0,1,1,3,23)\)

The only such pair is \([(x, y) = (223, 892)\]. For \(x\) and \(y\) positive the equation is equivalent to \(x + y + 2\sqrt{xy} = 2007\). This shows that \(\sqrt{xy}\) must be an integer, so if \(x = m^2a\) where \(a\) has no square factors larger than 1, then \(y\) must be of the form \(n^2a\), with \(0 < m \leq n\). Thus we have

\[2007 = x + y + 2\sqrt{xy} = m^2a + n^2a + 2\sqrt{m^2n^2a^2} = (m^2 + n^2 + 2mn)a = (m + n)^2a;
\]
i.e., \((m+n)^2a = 2007 = 3^2 \cdot 223 = 1^2 \cdot 2007\), which implies that \(m+n = 3\) (because \(m+n = 1\) is impossible), and \(a = 223\). With \(0 < x \leq y\), the only possibility is \(m = 1\), \(n = 2\), making \(x = m^2a = 223\) and \(y = n^2a = 4 \cdot 223 = 892\). As a check, we verify that

\[x + y + 2\sqrt{xy} = 223 + 4 \cdot 223 + 2\sqrt{223 \cdot 4 \cdot 223} = 9 \cdot 223 = 2007.\]
7. Difference of square roots. (6,0,0,0,0,0,1,0,0,46)

Yes. Upon expanding by the binomial theorem we obtain

\[(\sqrt{2007} - \sqrt{2006})^{2008} = a - b\sqrt{2007}\sqrt{2006},\]

where \(a\) and \(b\) are integers. Then

\[(\sqrt{2007} + \sqrt{2006})^{2008} = a + b\sqrt{2007}\sqrt{2006},\]

and


Let \(N = a^2\). Then \(N - 1 = b^2(2007)(2006)\), and

\[\sqrt{N} - \frac{1}{\sqrt{N}} = a - b\sqrt{2007}\sqrt{2006} = (\sqrt{2007} - \sqrt{2006})^{2008}.\]

8. Group product (7,0,0,0,0,0,0,0,3,5,38)

Consider the \(n + 1\) elements 1, \(a_1\), \((a_1a_2)\), \((a_1a_2a_3)\), \ldots, \((a_1a_2\cdots a_n)\). As \(G\) has just \(n\) elements, some two of these \(n + 1\) elements are equal. If \(1 = \prod_{k=1}^{s} a_k\) for some \(s\), we have the desired result. In the remaining case,

\[\prod_{k=1}^{t} a_k = \prod_{k=1}^{u} a_k\]

for some \(t\) and \(u\) with \(1 \leq t < u \leq n\). In this case,

\[\prod_{k=1}^{t} a_k = \prod_{k=1}^{t} a_k \prod_{k=t+1}^{u} a_k,\]

and we have

\[1 = \prod_{k=t+1}^{u} a_k,\]

completing the proof.
9. Limit of a sequence of integrals. \((1,2,0,0,2,0,0,1,4,0,43)\)

The limit is \(7/24\). We will use the fact that

\[
\frac{1}{(x+1)^4} < \frac{x}{x^5+1} < \frac{1}{x^4}
\]

for \(x \geq 1\),

which is evident upon clearing of fractions and expanding. It follows from (1) that

\[
I_n := n^3 \int_n^{2n} \frac{dx}{(x+1)^4} < n^3 \int_n^{2n} \frac{xdx}{x^5+1} < n^3 \int_n^{2n} \frac{dx}{x^4} =: J_n
\]

for every \(n \geq 1\). Now

\[
I_n = n^3 \left( \frac{1}{(n+1)^3} - \frac{1}{(2n+1)^3} \right) = \frac{1}{3} \left( \left( \frac{n}{n+1} \right)^3 - \left( \frac{n}{2n+1} \right)^3 \right) \to \frac{1}{3} \left( 1 - \frac{1}{8} \right) = \frac{7}{24},
\]

and

\[
J_n = n^3 \left( \frac{1}{n^3} - \frac{1}{(2n)^3} \right) = \frac{1}{3} \left( 1 - \frac{1}{8} \right) = \frac{7}{24}.
\]

It follows that

\[
\lim_{n \to \infty} n^3 \int_n^{2n} \frac{dx}{x^5+1} = \frac{7}{24}.
\]

10. Lattice points on a curve. \((1,0,0,0,1,0,0,3,1,1,46)\)

Suppose \((x, y)\) is a lattice point on the given curve. From the facts that

\[
\left( \frac{x^2 + x}{2} \right)^2 = x^4 + x^3 + \frac{x^2}{4} = y^2 - \frac{3}{4} x^2 - x - 1 = y^2 - \frac{3}{4} \left( x + \frac{2}{3} \right)^2 - \frac{2}{3} < y^2,
\]

and

\[
(x^2 + \frac{x}{2} + 1)^2 = x^4 + x^3 + \frac{9}{4} x^2 + x + 1 = y^2 + \frac{5}{4} x^2 \geq y^2,
\]

we conclude that

\[
x^2 + \frac{x}{2} < |y| \leq x^2 + \frac{x}{2} + 1. \tag{2}
\]

If \(x\) is odd, then \(|y| = x^2 + (x+1)/2\) is the only integer in this interval, and

\[
y^2 = \left( x^2 + \frac{x + 1}{2} \right)^2 = x^4 + x^3 + \frac{x^2 + 2x + 1}{4}
\]

\[
= x^4 + x^3 + \frac{x^2}{2} + x + 1 + \frac{1}{4} (x^2 - 2x - 3) = y^2 + \frac{1}{4} (x - 3)(x + 1).
\]

It follows that \((x - 3)(x + 1) = 0\), and so \(x = 3\) or \(x = -1\). This gives us the lattice points \((3, \pm 11)\) and \((-1, \pm 1)\). If \(x\) is even, then (2) implies that \(|y| = x^2 + x/2 + 1\). Then (1) implies that \(y^2 = y^2 + (5/4)x^2\), and therefore that \(x = 0\), giving us the lattice points \((0, \pm 1)\), and so these are the only solutions.