Research Description

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Overview and Background. I work in the field of algebraic combinatorics with a focus on combinatorial models of representations of Lie algebras. This pertains mainly to the use of combinatorial structures to encode complex algebraic objects and then make computations and conclusions using combinatorial techniques. The idea is that combinatorial models are easier to manipulate than the original algebraic object, but contain all of the structure. In research pertaining to these combinatorial models, it is often useful to write software to encode the structure. Such software is used in many stages of research: the development of the combinatorial model, testing to make conjectures, and even to give insight as to how to prove those conjectures. **Here, I will describe my research interests and future work and then present plans for current and future projects including students.**

My research has, by and large, been threefold: one part, pertaining to Kirillov-Reshetikhin modules and the quantum alcove model, a second, pertaining to infinity crystals realized through the theory of PBW bases, and a third, pertaining to Kostka-Foulkes polynomials. Underlying all of these projects are *Lusztig q-analogue polynomials* and the use of the combinatorial structure of *crystals*. **Numbered references are those which I have co-authored.**

Crystal bases were independently discovered by Kashiawara and Lusztig in the early 1990's as a means of deciphering nice combinatorial properties of representations of quantum groups (noncommutative analogs of Lie groups) and are the result of taking the limit of the quantum group's parameter q to zero. Not only do these crystals have very nice combinatorial structure, as they are colored directed graphs, but they also encapsulate all of the structure of the corresponding representation. Each representation $V(\lambda)$ for a highest weight λ of a semisimple Lie algebra has a corresponding crystal denoted $B(\lambda)$. The edges of these graphs are colored by *crystal operators* which are related to the corresponding Chevalley generators. There are many combinatorial realizations of these crystal graphs. The classical model uses semistandard Young tableaux for its vertices and a simple so-called bracketing rule to compute the crystal operators for the edges.

Kirillov-Reshetikhin (KR) crystals correspond to certain finite-dimensional representations of untwisted affine Lie algebras. There are two main combinatorial models for tensor products of (column shape) KR crystals: the tableau model (type specific, applying to the classical types) and the quantum alcove model (uniform for all untwisted affine types). While the tableau model is simpler, it has less easily accessible information so it is generally hard to use in specific computations: of the energy function (defining a grading on KR crystals), the combinatorial R-matrix (the unique affine crystal isomorphism commuting tensor factors), etc. As these computations are much simpler in the quantum alcove model, an alternative is to relate them to the tableau model via an affine crystal isomorphism. This has been achieved in types A and C by Lenart in [Len, '12],[LL, '15b].

- In [1] Briggs, Lenart and I were able to take the first step in the direction of extending this work to types B and D by providing an explicit bijection between the classic tableau model of Kirillov-Reshetikhin crystals and the quantum alcove model. Due to key differences in the root systems and *quantum Bruhat graphs* (a version of the classical Bruhat graph with additional edges corresponding to certain decreases in Weyl group length), there were many additional features in these new bijections.
  - I have been able to present this work as a poster at the 2019 AlgeCom (Algebra, Geometry and Combinatorics) conference in Chicago,
  - as well as in a talk given at the 2019 Graduate Student Combinatorics Conference in Philadelphia.
  - I am currently working on extending these bijections to explicit crystal isomorphisms.
  - This would then give way to more projects, such as calculating the energy function for these crystals
  - as well as providing a model for the combinatorial R-matrix in types B and D, extending the work in [LS, '11] and [LL, '15a] respectively.

- Further, plans have been made to extend this to the more general *rectangular shape* Kirillov-Reshetikhin crystals in joint work with T. Scrimshaw and his work with Rigged configurations.
- **A discussion below describes the ease at which this work can include undergraduate involvement at a diversity of mathematical levels and disciplines.**

Infinity Crystals. The infinity crystal $B(\infty)$ is defined as the direct limit of all irreducible highest weight crystals $B(\lambda)$ of a given Cartan type. They are the crystals associated with Verma modules of highest weight zero corresponding to a symmetrizable Kac-Moody algebra and are an infinite dimensional counterpart to the $B(\lambda)$ described above.

Lusztig's early construction of crystal basis can be interpreted as giving a number of parameterizations of $B(\infty)$, one for each reduced expression for the longest element in the Weyl group [Ting, '17]. In each of these realizations at least one of the crystal operators is very simple, but others may be complicated. However, Lusztig explicitly describes how the realizations are related for reduced expressions that differ by a braid move. This provides a means of realizing the whole crystal: an element is a PBW monomial with respect to some chosen reduced expression. To apply a crystal operator, modify the element via a sequence of braid moves until that operator is simple, then apply the operator, then modify it back. This procedure is algorithmic, but can be complicated. In type A_n there is a simpler realization, using multisegments, where the crystal operators are given by a simple bracketing rule [CT, '15].

- In [5], we generalize this by giving conditions on a reduced expression that ensure Lusztig's crystal structure is given by a similar rule and then describe the resulting structure. For these words the crystal operators can be understood by combining many rank two calculations, and this combining is controlled by a bracketing procedure. There is at least one such reduced expression in every classical type.
  - This work was presented as a poster at the 2017 Formal Power Series and Algebraic Combinatorics conference in Vancouver.
- The code co-written with Peter Tingley and Dinakar Muthiah implementing this crystal structure has been included in the Sagemath library [7].
- In [6] we give an explicit description of the unique crystal isomorphism between two realizations of $B(\infty)$; that using marginally large tableaux and that using PBW monomials with respect to one particularly nice reduced expression of the longest word.

Lusztig t-analogues and Charge. Lusztig defined a polynomial $K_{\lambda,\mu}(t)$ as a $t-analogue$ of the multiplicity of μ in the representation $V(\lambda)$ for highest weight λ of a semisimple Lie algebra. These are commonly referred to as Kostka-Foulkes polynomials and they have very nice properties. They are special affine Kazhdan-Lusztig polynomials and are related to Hall-Littlewood polynomials (spherical functions for a Chevalley group over a p-adic field). Further, they are closely related to the energy function for exactly solvable (or integrable) lattice models coming from statistical mechanics.

It has long been known that the Kostka-Foulkes polynomial can be written with all non-negative coefficients. A statistic on words with partition content, called *charge*, was used to give a combinatorial formula demonstrating this fact in type A. Defining a charge statistic beyond type A has been a long-standing problem. We briefly look at two recent developments which provide another base for my current research.

I: Lascoux gave another combinatorial formula to show the non-negativity of Kostka-Foulkes polynomials in type A by decomposing them into so-called atomic polynomials. Unfortunately, the complexity of the decomposition made it difficult to extend to other Lie types. Recently, Lenart and Lecouvey gave a new model for such an atomic decomposition with a much simpler combinatorial structure, named *modified crystals* [LL, '18], allowing them to extend beyond type A.

- During a stay at the Université de Tours in June of 2018, I worked under Lecouvey to map out the so-called *small intervals of the dominance order* in types C and D, a key aspect of the modified crystals.

In [LL, '18], they explicitly described this decomposition in all classical finite types for the stable cases (meaning for sufficiently large n). This decomposition of Kostka-Foulkes polynomials gave way to a statistic that matched the classic charge statistic in type A. However, it has proven difficult to relate the atomic decomposition to the charge statistic beyond type A.

- During an extended stay at Université de Tours in 2019, working as a Chateaubriand Fellow, I began work with Lecouvey to extend this atomic decomposition to untwisted affine types.

II: In [4], we take a completely different approach to extending the charge beyond Type A by considering the definition of Kostka-Foulkes polynomials as an alternating sum over Kostant partitions as opposed to the previous models which are based on tableaux. We use the structure of the crystals $B(\infty)$ and $B^*(\infty)$ (given in [CT, '15] and [5]) to construct a collection of modified crystal operators on Kostant partitions. We then provide a matching on the resulting crystal graph that corresponds to a sign changing involution which cancels out all of the negative terms in the Kostka-Foulkes polynomial. This positive expansion in terms of Kostant partitions gives way to a statistic which is simply read by just counting the number of parts of the Kostant partitions. The hope is that the simplicity of this new crystal-like model will naturally extend to other classical types.

- I am currently making plans to work on extending this to type D using results from [6] (and then to types B and C) with undergraduate involvement.

References

[1] C. Briggs, C. Lenart, **A. Schultze**. On Combinatorial Models for Affine Crystals. ArXiv:2109.12199.

[2] A. Engbrecht, M. Liu, A. Rajan, **A. Schultze**. A Special Case of a Charge Statistic Beyond Type A. In preparation.

[3] S. Gama, **A. Schultze**. On Kirrilov Reshethikin Crystals of Types $B_n^{(1)}$ and $D_n^{(1)}$. In preparation.

[4] C. Lecouvey, C. Lenart, **A. Schultze**. Towards a Combinatorial Model for q-weight Multiplicities of Simple Lie Algebras. ArXiv:2110.15394.

[5] B. Salisbury, **A. Schultze**, and P. Tingley. Combinatorial descriptions of the crystal structure on certain PBW bases. *Transform. Groups*, 2018.

[6] B. Salisbury, **A. Schultze**, and P. Tingley. PBW bases and marginally large tableaux in type D. *J. Comb.*, 9: 535 – 551, 2018.

[7] D. Muthiah, **A. Schultze**, P. Tingley, Implementation of the crystal of PBW monomials and MV polytopes in Sage, *SageMath*, https://trac.sagemath.org/ticket/18655.

[CT, '15] J. Claxton and P. Tingley. Young tableaux, multisegments, and PBW bases. *S´em. Lothar. Combin.*, 73, Article B73c, 2015.

[CCHJR, '20] J. Carlson, A. Christensen, P. Harris, Z. Jones, A.R. Rodríguez. Parking Functions: Choose Your Own Adventure. arXiv:2001.04817, 2020.

[LL, '18] C. Lecouvey, C. Lenart. Atomic decomposition of characters and crystals. *arXiv:1809.01262v1* , 2018.

[Len, '12] C. Lenart. From Macdonald polynomials to a charge statistic beyond type A. *J. Combin. Theory Ser. A*, 119:683-712, 2012.

[LL, '15a] C. Lenart, A. Lubovsky. A uniform realization of the combinatorial R-matrix. *Adv.in Math.*, 334:151-183, 2015.

[LL, '15b] C. Lenart, A. Lubovsky. A generalization of the alcove model and its applications. *J. Algebraic Combin.*, 41:751-783, 2015.

[LNSSS, '17] C. Lenart, S. Naito, D. Sagaki, A. Schilling, M. Shimozono. A uniform model for Kirillov-Reshetikhin crystals II. Alcove model, path model, and P=X. *Int. Math. Res. Not.* 2017, no. 14, 4259–4319.

[LS, '11] C. Lenart and A. Schilling. Crystal energy functions via the charge in types A and C. *Math. Z., DOI* 10.1007/s00209-012-1011-2.

[Ting, '17] P. Tingley. Elementary construction of Lusztig's canonical basis, Groups, Rings, Group Rings and Hopf Algebras. *Contemporary Mathematics.* 688:265–277, 2017.