Research Talk

Algebraic Combinatorics and the math of numbers in boxes

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Lewis University, December 14^{th}, 2022

December 12, 2022

What is Algebraic Combinatorics?

Algebra

- Study of Sets
  - and how their elements interact with each other

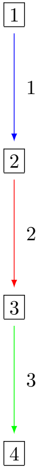
- Jargen Alert!
  - Representations of Lie Algebras
  - change of bases of certain sets of matrices
  - applications in physics

Combinatorics

- Convert into *puzzles*
  - encode the algebra into something easier

- Main Puzzle!
  - Crystal Graphs

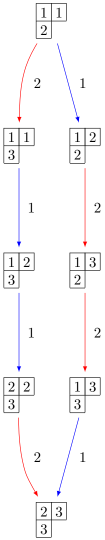
A Crystal Graph Example



Notes:

- Numbers in boxes called **crystal vertices**
- Colored arrows called **crystal edges**
- Always have $\boxed{i} \xrightarrow{i} \boxed{\text{i+1}}$

Another Crystal Graph Example



Notes:

- Lower numbers: only goes up to 3 now
- More boxes! Has *shape* $[2, 1]$
- Still always have $\boxed{i} \xrightarrow{i} \boxed{\text{i+1}}$

How do we make these crystals!

Two Ingredients

Pick an integer $n > 1$ and $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{n-1})$

Three Algorithms

Step I:

We **create a shape** using λ

Step II:

If we **fill these diagrams** with numbers from 1 to n

Step III:

Make the edges to change a number i to an $i + 1$.

Make the Shape

Step I:

We create a shape called a Young diagram:

1. glue together rows of boxes,
2. one for each integer in λ,
3. from tallest to smallest
4. so they are upper left justified.

Filling the diagrams

Step II:

We fill these diagrams with numbers from 1 to n

1. columns are strictly increasing
2. rows are weakly increasing
3. we get what we call a *semi-standard Young tableaux*

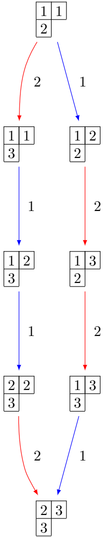
These are the vertices of the crystal!

Calculating Crystal Edges!

Step III:

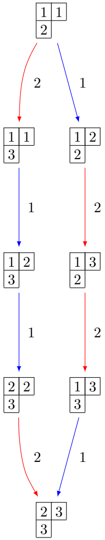
The edges change a number i to an $i+1$. But which one?!

1. write out the values of the tableux starting from the bottom row to the top, reading rows from left to right,
2. place a $+$ above every i and a $-$ above every $i+1$,
3. cancel all adjacent $-$ and $+$ pairs,
4. change the i corresponding to the right-most remaining $+$ to an $i+1$. If no such $+$ exists, then the tableau is not the source vertex of an arrow labeled i.
5. The resulting tableau is where the arrow points!



Notes:

- The choice of n
- The shape of the vertices
- Filling the vertices with numbers
- Defining the edges



change of bases of certain sets of matrices

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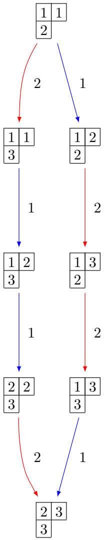
Crystal Graphs - with math jargon

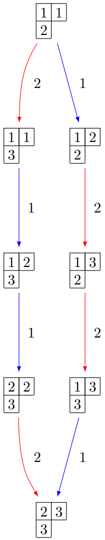


change of bases of certain sets of matrices

Notes:

- The choice of n
  - Comes from the use of $n \times n$ matrices
- The shape of the vertices
- Filling the vertices with numbers
- Defining the edges

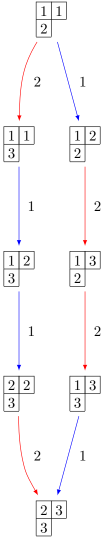
Crystal Graphs - with math jargon



change of bases of certain sets of matrices

Notes:

- The choice of n
  - Comes from the use of $n \times n$ matrices
- The shape of the vertices
  - Corresponds to the change of basis
- Filling the vertices with numbers
- Defining the edges



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  - Comes from the use of $n \times n$ matrices
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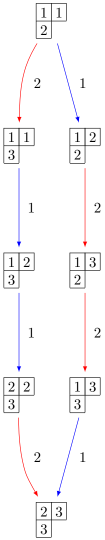
  - Comes from certain eigenspaces of the matrices
- Defining the edges



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- The choice of n
  - Comes from the use of $n \times n$ matrices
- The shape of the vertices
  - Corresponds to the change of basis
- Filling the vertices with numbers
  - Comes from certain eigenspaces of the matrices
- Defining the edges
  - Comes from the interaction between the matrices through multiplication

Thank you!

Student Involvement

Two recent student projects...

- Giving a map between two models of crystals: Tableau and Quantum Alcove
  - One of my calc II students
  - Can be done purely combinatorially/algorithmically
  - The map *extends* columns in tableaux.
  - Student showed that the crystal arrows are independant of the extending.
- Modifying crystals to answer other algebra problems
  - Three upper level students
  - Each approaching from different perspectives: computer science, linear algebra, abstract algebra.

The second project

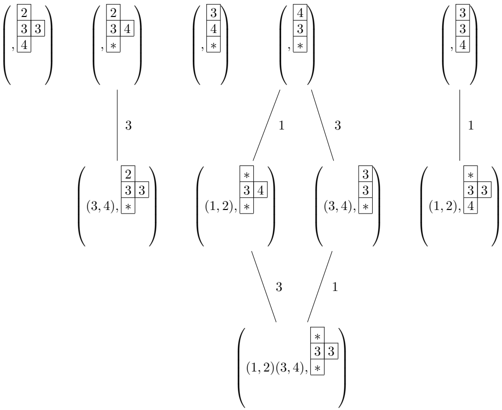

Figure: Modifying crystals to simplify so-called Kostka–Foulkes polynomials

Thank you!